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AN
INTERMEDIATE
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BY

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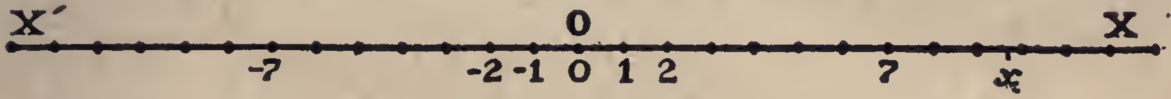
CHAPTER I

EQUATIONS

1. **Preliminary.** In the expression $2x - 3$ appear the numbers $2, -3, x$. Of these, $2, -3$, having definite values which cannot change, are called **constants,** while the number x, regarded as capable of assuming different values, is called a **variable.** In general, it is supposed that the variable x may assume, or pass through, all assignable positive or negative values, as well as the value zero. Numbers generally present themselves as measuring some physical or geometrical quantity, the constant measuring a certain definite quantity and the variable measuring a quantity undergoing change. On this account numbers and expressions involving numbers are frequently spoken of as quantities. It is to be noted, too, that, in the case of variable quantities, the physical or geometrical conditions may be such as to limit the range of variation of the variable.

A convenient and useful representation of number, whether constant or variable, is found in the straight line regarded as indefinitely produced each way. A point O is taken, in this line, as origin, or place from which measurements are made.



Each point on this line, being at a certain distance from the origin, can be looked upon as carrying a certain number, namely, the measure of the distance of that point from the origin. It is agreed that positive numbers are to be carried by points to the right of the origin, and negative numbers by points to the left, while the number zero is carried by the point O, the origin. A fixed point on the line represents a constant number; a point, moving or regarded as capable of moving in the line, a variable number.

Return now to the expression $2x - 3$. If to x be assigned the values

$$0,\ \tfrac{1}{2},\ 1,\ 1\tfrac{1}{2},\ 2,\ \ldots\ldots$$

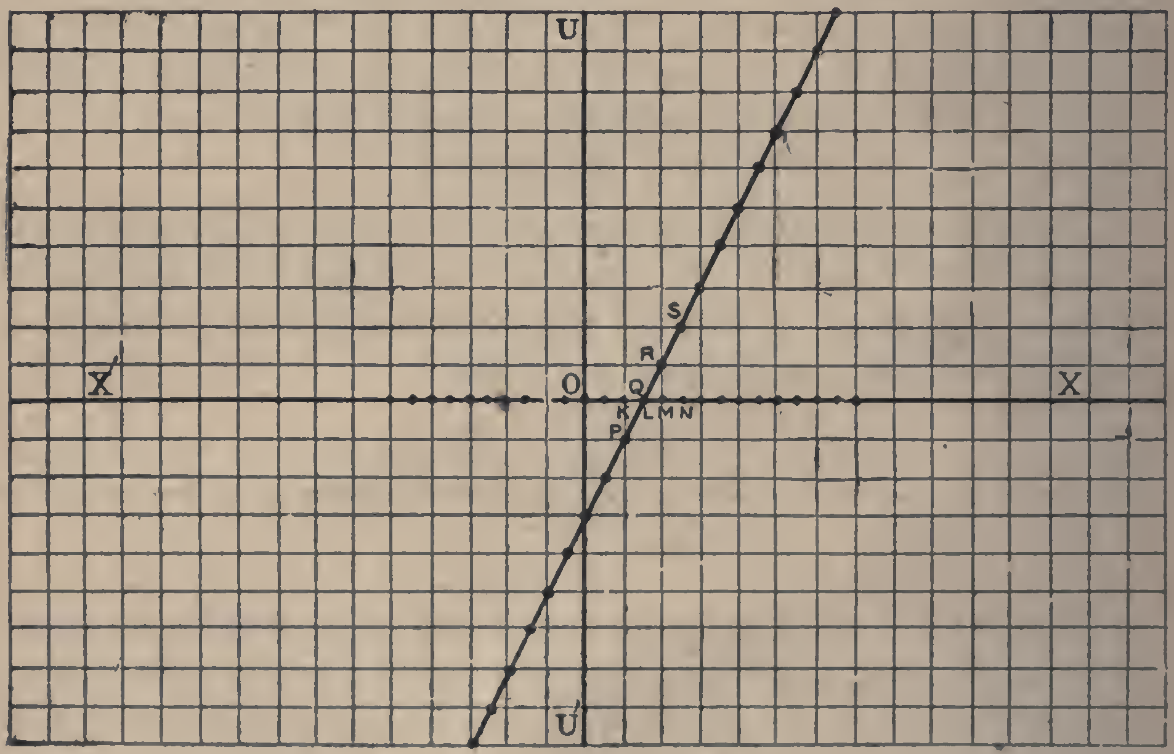
the values of $2x - 3$ are found to be

$$-3,\ -2,\ -1,\ 0,\ 1,\ \ldots\ldots$$

It follows then that $2x - 3$ is also a variable.

Any quantity whose value is determined when the value of a certain variable quantity, as x, is given, is called a **function** of that variable. Thus, $2x - 3$ is a function of x. This function is further described as **one-dimensional,** or **linear** in the variable x.

As has been seen, the linear function $2x - 3$ is a variable, its value depending on and changing with that of x. On this account x is called the **independent variable,** and if the value of the function $2x - 3$ is denoted by u, then u is called the **dependent variable.** The relation of u to x can be exhibited by means of a diagram. First, draw X′OX on which to represent the values of x, O being the origin. Then through O draw UOU′ at right angles to this line.



First, suppose x to have the value $2\frac{1}{2}$, carried by the point N. The corresponding value of u is found to be 2. Mark the point S at a distance 2 from N in the direction OU. Then this point S indicates by its distance from the point N in OX, just below it, the value of u for $x = 2\frac{1}{2}$, the value carried by N.

Next, for $x = 2$, carried by the point M, the value of u is 1, and, as before, the point R indicates by its position that the value of u is 1 if $x = 2$.

So for $x = 1\frac{1}{2}$, carried by the point L, the value of u is 0, and the point Q (the same point as L) on the line X′OX indicates this fact.

For $x = 1$, carried by the point K, the value of u is -1. We agree to indicate negative values of u by measurements made below the line X′OX, so that the point P indicates that $u = -1$ for $x = 1$.

Thus, to each value of x corresponds a value of u, and if we construct the assemblage of points PQRS ..... giving the values of u corresponding to the values of x belonging to all the points on X′OX, a continuous line PQRS ..... will be formed. Here this continuous line is found to be a straight line. It is said to be the **graph** of the function $2x - 3$, and, as has been explained, it puts in evidence the value of the function for any or all values of x considered. The lines X′OX, UOU′, to which the measurements determining the graph have been referred, are called the **axes.**

Consider now the expression $ax + b$. Here it is supposed that a and b are constants, and that x is a variable, so that $ax + b$ is a linear function of x. The numbers a and b may have any values, but, whatever be these values, they are constant. The expression $ax + b$ is thus the **general linear function** of x. For any given values of a and b the graph of $ax + b$ may be constructed, and can be seen to be a straight line.

An expression as

$$\frac{2x - 3}{3x - 5}$$

in accordance with what has been stated, is a function of x. It is also a linear function of x as in it x appears in the first and no higher degree. Here, however, x occurs in the denominator, or, in other words, the function is fractional in its relation to x. On this account it is said to be a **fractional linear function** of x, while a function as $ax + b$ is said to be an **integral linear function.**

EXERCISES

1. Construct the graph of the following linear functions:

$$x-7,\ 3x,\ 2x+4,\ 6-8x.$$

2. Construct the graph of

$$\frac{x+3}{2x-1}$$

for values of x in the interval $(-2, +2)$.

3. Shew that, whatever be the values of a and b, the graph of $ax+b$ is a straight line.

Hence shew that when two values of any integral linear function have been found the graph may be constructed.

2. **Significance of an Equation.** Suppose that the solution of the following simple problem is required:

Find the number, the double of which diminished by 3 is equal to zero.

Let x denote the number. Then by the condition given

$$2x-3=0.$$

We have here an **equation,** a statement to the effect that the expression $2x-3$ *is equal to* zero. It is readily seen that

$$2x=3 \text{ and } x=1\tfrac{1}{2}.$$

The rôle of the equation is now manifest. If it had been a question of the *expression* or *function* $2x-3$ only, we should have regarded x as a variable, and as x varied the value of $2x-3$ also would have varied. The *equation* declares that $2x-3$ is not to vary but to have the definite value zero, and it follows that x is not a variable but has a determinate value $1\frac{1}{2}$. In an equation then, as the one just treated, the x, which is not a variable but a definite number whose value is sought, will be called the **unknown quantity,** or simply the **unknown.** The value of the unknown is called the **root** of the equation.

The relation of the *expression* $2x-3$ to the *equation* $2x-3=0$ is now clear. For different values of x the expression takes different values; the equation makes it impossible for x to vary, and we are required to find the invariable or definite value of x which will make $2x-3$ equal to zero. In like manner we could find the value of x that would make $2x-3$ equal to any given number. The graph constructed in the preceding Art. puts this in evidence, as well as the fact that when $2x-3=0$, the value of x is $1\frac{1}{2}$.

Not infrequently there present themselves equalities which are not equations like the one considered, in that they do not require that x should have a definite value, but hold for all values of x. Such, for example, is the relation

$$(x+1)(x-1)-(x^2-1)=0$$

or, which is the same thing,

$$(x+1)(x-1)=x^2-1.$$

If the implied multiplication is performed it is seen that the equality holds whatever be the value of x, or, in other words, that $(x+1)(x-1)$ and x^2-1 are two *different forms of one expression.* Such an equality is generally called an **identical equation** or an **identity,** so that when the term equation is employed it is to be understood in the sense first explained.

EXERCISES

1. Construct the equations pertaining to the following problems :

   (1) Find two consecutive integers such that their product will exceed the square of the less by 13.

   (2) A father is three times as old as his son, but in 15 years he will be only twice as old as his son. Find their present ages.

   (3) A can run 100 yd. in 10 sec., and B in 11 sec. They start together in a race of 100 yd. At the end of what time will A be midway between B and the winning post?

2. Shew that the following problems lead to identities, stating in each case the inference to be drawn from this fact :

   (1) Find three consecutive integers such that the product of the greatest and the least is less than the square of the mean integer by unity.

   (2) Divide a straight line of given length a (units), so that the square on the given line may be equal to the squares on the two parts, increased by twice the rectangle contained by the parts.

3. **The Simple Equation.** The general linear equation is

$$ax+b=0.$$

It is supposed that a is not zero; for if this were the case, then would ax equal zero, and, therefore, also b equal zero, and there would be no equation.

It follows at once that

$$ax=-b$$

and, therefore, that

$$x=-\frac{b}{a},$$

since, a not being zero, division is possible.

These results are necessary and it follows that:

Every linear equation has one and only one root.

It is well to suppose any such equation to have originated in some actual problem, x measuring some quantity whose magnitude is sought.

An equation of the form

$$ax+b=cx+d,$$

is readily seen to be not more general than the equation $ax+b=0$.

EXERCISES

1. Solve the equations:

(1) $\frac{2x-7}{5}+\frac{x-5}{6}=\frac{x+3}{7}+\frac{x+7}{9}.$

(2) $5x-\left(\frac{x}{2}-\frac{x}{3}\right)=7\left(\frac{x}{3}+\frac{x}{4}\right)+9.$

(3) $\frac{x}{p}+\frac{x}{q}+\frac{x}{r}=qr+rp+pq.$

(4) $\frac{x}{bc}+\frac{x}{ca}+\frac{x}{ab}=a+b+c.$

✓ 2. The sides of a triangle are 39 ft. 42 ft., 45 ft. in length. A perpendicular is drawn to the side whose length is 42 ft. from the opposite angle. Find the length of the segments into which this side is divided.

Construct the triangle according to scale, draw the perpendicular as indicated, and measure the segments to test the accuracy of the *drawing*.

4. The Quadratic Expression. Consider the expression

$$x^2 - 4x + 3.$$

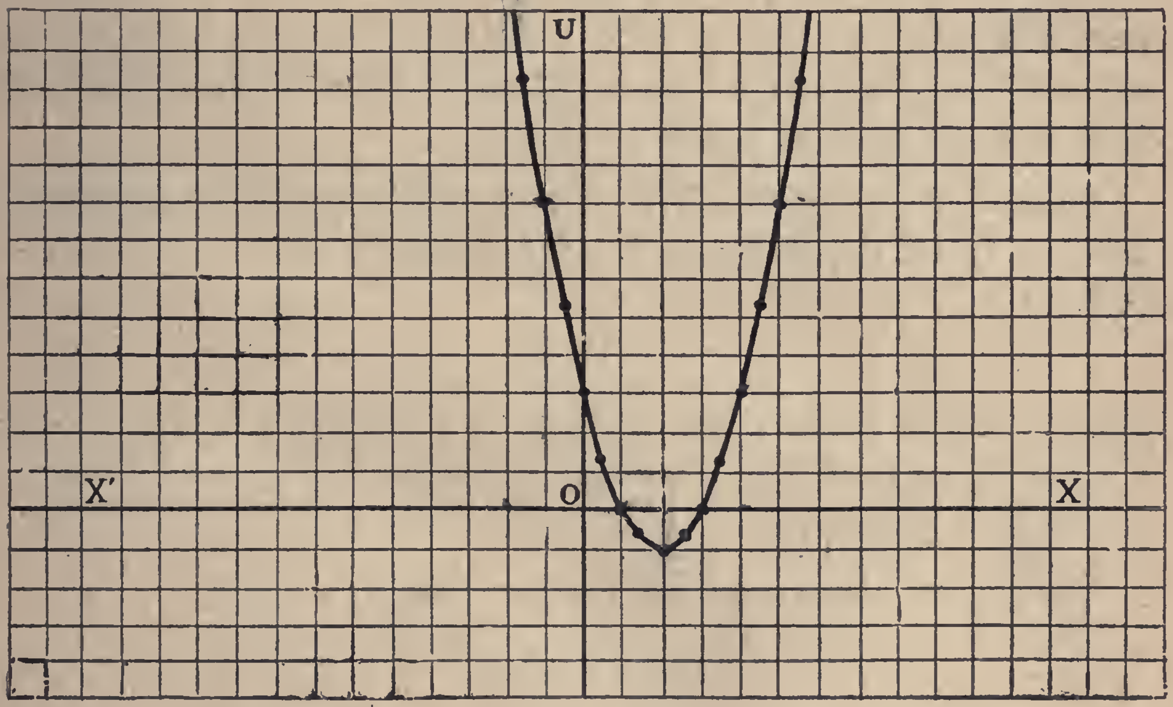
In this expression, the numbers -4, 3 are constants and the number x a variable. To each value of x corresponds a value of the expression. Thus, if x is assigned the values

$$-1,\ 0,\ 1,\ 2, \ldots\ldots$$

the values of $x^2 - 4x + 3$ are found to be

$$8,\ 3,\ 0,\ -1, \ldots\ldots$$

The expression $x^2 - 4x + 3$, for reasons already given, is then a function of x. It is further described as a **function of the second degree,** or a **quadratic function** as in it x occurs to the second power, and not to any higher power. The function is integral since x does not appear as a denominator or as making part of a denominator. If the value of the function is denoted by u, then u is a variable, and a graph may be constructed which will exhibit the dependence of u upon x.



The graph is not a straight line, as in the case of the linear function, but a curve which is called a **parabola.** It is to be noted that the curve is symmetrical about a line at right angles to the axis X′OX through the point carrying the value 2 of x. The reason for this may be shewn, for we have

$$\begin{aligned} u &= x^2 - 4x + 3 \\ &= x^2 - 4x + 4 - 1 \\ &= (x-2)^2 - 1. \end{aligned}$$

Now, first let $x = 2 - k$, a value k less than, or behind 2.

Then for this value of x

$$\begin{aligned} u &= (2 - k - 2)^2 - 1 \\ &= k^2 - 1. \end{aligned}$$

Next, let $u = 2 + k$, a value k greater than, or in advance of 2.

Then for this value of x

$$\begin{aligned} u &= (2 + k - 2)^2 - 1 \\ &= k^2 - 1. \end{aligned}$$

Therefore, whatever be the value of k, these two values of u are the same, and hence any two values of x equidistant from 2 give the same value of u, or, in other words, the graph is symmetrical about a line through $x = 2$ at right angles to X′OX.

Further, the graph indicates that the smallest value, or **minimum,** of u is -1, and this when $x = 2$. The reason for this is manifest, for we have

$$u = (x-2)^2 - 1.$$

Now, $(x-2)^2$ is positive for all values of x except $x = 2$, when $(x-2)^2$ is equal to zero. When, therefore, x is equal to 2, the value of u is the smallest possible.

The general linear quadratic function may be written

$$ax^2 + bx + c$$

and for any assigned values of the constants a, b, c, the behaviour of the function under the variation of x may be studied, and indicated by a diagram.

EXERCISES

1. Construct the graphs of the following functions :

$$x^2,\ x^2-1,\ x^2-x+6,\ x^2-2x+3.$$

In each case find the minimum of the function and the value of x which gives this minimum.

2. Construct the graphs of the following functions :

$$-x^2+x-6,\ -x^2,\ -x^2+1,\ 3+2x-x^2.$$

In each case find the maximum of the function and the value of x which gives this maximum.

3. Construct the graphs of the following functions :

$$2x^2-3x-5,\ 3x^2-5x+7,\ 4+5x-2x^2.$$

In each case find the minimum or maximum of the function and the corresponding value of x.

4. Construct the graph of the fractional quadratic function

$$\frac{x^2-x-2}{x^2-5x+6}$$

for values of x between -2 and $+4$.

5. **Quadratic Equations.** Suppose that the solution of the following problem is required:

Divide the number 4 into two parts such that the sum of the parts exceeds their product by 1.

If one part be the number x, then the other part is $4-x$.

The sum of the parts $=x+(4-x)=4$.

The product of the parts $=x\ (4-x)$.

$$\begin{aligned} \therefore\ 4\ -x\ (4-x) &= 1 \\ \therefore\ x^2-4x+4 &= 1 \\ \therefore\ x^2-4x+3 &= 0. \end{aligned}$$

In this **quadratic equation,** x is presumably not a variable but some definite number, an unknown as yet. We are to seek the value of x in this equation, or, in other words, to solve the equation.

If now

$$x^2 - 4x + 3 = 0$$

then must

$$(x-3)(x-1) = 0$$

and conversely. This last relation is satisfied if either

(1) $x - 3 = 0$, *i.e.*, $x = 3$; for then $x - 3 = 0$ and $x - 1 = 2$, so that the product $(x-3)(x-1)$ is equal to zero,

or

(2) $x - 1 = 0$, *i.e.*, $x = 1$, for then $x - 1 = 0$, and $x - 3 = -2$, so that the product $(x-3)(x-1)$ is equal to zero.

Thus the equation has *two roots,* namely 3 and 1, and it follows that x does not represent a definite number but one of two definite numbers. This is a result that might have been expected, since, x standing for one of the two parts, and the equation having been formed on this supposition, each part has an equal claim to recognition.

The root $x = 3$ gives 3 and 1 as the two parts of 4, and the root $x = 1$ gives 1 and 3 as those parts, so that the two solutions refer to one mode of division, the parts being mentioned in different orders.

Further, no value of x other than 3 and 1 will satisfy the equation, for any such value would make neither $x - 3$ nor $x - 1$ equal to zero and the product $(x-3)(x-1)$ could not then be zero. Thus, this quadratic equation has two and only two roots. It is to be noted, too, that the solution of this quadratic equation consists in replacing it by *two linear equations.*

The graph of the function given in Art. 3 brings out the fact that for $x = 1$ or $x = 3$ the value of the function $x^2 - 4x + 3$ is zero.

As in the case of the linear equation it is well to regard any proposed quadratic equation as having originated in some problem, x denoting some unknown number whose value is sought.

We shall now study the general quadratic equation with a view to discover general properties of such equations. Let the general quadratic equation be

$$ax^2 + bx + c = 0$$

where a is supposed not to be zero.

Then must

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0$$

$$\therefore a\left[\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)\right]=0$$

$$\therefore a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]=0$$

$$\therefore a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)=0$$

$$\text{or } a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)=0.$$

This last equation is then satisfied by those values of x which satisfy the proposed equation and conversely. Now, in order that this last equation may be satisfied either

(1) $x+\dfrac{b+\sqrt{b^2-4ac}}{2a}$ *must* equal zero, which requires that

$$x=\frac{-b-\sqrt{b^2-4ac},}{2a}$$

or

2) $x+\dfrac{b-\sqrt{b^2-4ac}}{2a}$ *must* equal zero, which requires that

$$x=\frac{-b+\sqrt{b^2-4ac},}{2a}$$

Plainly any value of x other than these will give to the factors on the left a value different from zero, so that the expression on the left cannot vanish. Accordingly,

The general quadratic equation has two and only two roots.

From the solution above given it is plain that the *factoring of the expression* ax^2+bx+c, and the *solving of the equation* $ax^2+bx+c=0$ are equivalent problems. The quadratic expression can always be presented as the product of two factors linear in x and, it may be, an additional factor not involving x; the quadratic equation is solved by replacing it by two linear equations. This relationship of the expression and the equation may be brought out otherwise, as follows.

Denote the roots of the equation $ax^2+bx+c=0$ by m and n so that, say,

$$\left.\begin{aligned} m &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\ n &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned}\right\} \text{I.}$$

These roots are algebraically irrational, that is, they require for their expression root symbols, though it may be that for given values of a, b, c, the arithmetical values of the roots may be found. If the roots are added and multiplied we have

$$\left.\begin{aligned} m+n &= -\frac{b}{a} \\ mn &= \frac{c}{a} \end{aligned}\right\} \text{II.}$$

The relations I and II are equivalent; in I the values of the roots are explicitly given, but the results given in II, on account of their simple form, are often more useful. We now have the expression

$$ax^2+bx+c=a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}, \text{ (identically.)}$$

$$=a\left\{x^2-(m+n)x+mn\right\},$$

by II, m and n denoting the roots of the *equation* $ax^2+bx+c=0$.

$$\therefore\ ax^2+bx+c=a(x-m)(x-n)$$

as seen before.

The two roots of the general equation are, in general, different, the difference being

$$m-n=\frac{\sqrt{b^2-4ac}}{a}.$$

This difference will be zero, *i.e.*, the two roots will be equal should it be that $\sqrt{b^2-4ac}=0$, *i.e.*, $b^2-4ac=0$ or $b^2=4ac$.

In this case the solution of the equation would have been as follows:

$$ax^2+bx+c=0.$$

$$\therefore\ a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}=0$$

$$\therefore\ a\left\{\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-\left(\frac{b^2-4ac}{4a^2}\right)\right\}=0$$

$$\therefore\ a\left\{x+\frac{b}{2a}\right\}^2=0,\text{ since } b^2-4ac=0.$$

The expression ax^2+bx+c is then a perfect square, except for the constant factor a, *i.e.*, is a perfect square in its relation to x, and the equation $ax^2+bx+c=0$ has *two equal roots,* each being $-\frac{b}{2a}$.

The solution of the quadratic equation leads to two remarkable extensions of the number concept. The numbers first employed were those that came to be called integers, as 1, 2, 3, ... The operation of division, arising through the solution of the linear equation, led to fractional numbers, as $\frac{2}{3}$, $\frac{3}{4}$, ... Also, algebra, dealing with general numbers, led naturally to the introduction of negative numbers as distinguished from positive numbers. We are now led further to extend the domain of number.

For first, consider the equation,

$$x^2+3x-1=0.$$

The roots are found to be

$$\frac{-3+\sqrt{13}}{2}\text{ and }\frac{-3-\sqrt{13}}{2}.$$

The square root of 13 cannot be exactly found, *i.e.*, cannot be expressed as a positive or negative integer or fraction, though it may be found to any degree of accuracy. Also, a line or other magnitude can be constructed which would have for measure the square root of 13. We are thus led to call $\sqrt{13}$ a number, an **irrational number,** in contrast to positive or negative integers or fractions which are **rational numbers.** Indeed, it is only after admitting the square root of 13 to be a number that we can say that the equation $x^2+3x-1=0$ has two roots.

Next, consider the equation

$$x^2 - 2x + 5 = 0.$$

The roots are found to be

$$\frac{2 \pm \sqrt{-16}}{2}.$$

The symbol $\sqrt{-16}$ is a symbol of something for which no equivalent exists among the numbers already treated, for the square of any positive number or of any negative number is a positive number, and, therefore, cannot equal -16. After the analogy of the earlier extensions, we shall call $\sqrt{-16}$ a number, an **imaginary number,** to distinguish it from those numbers previously considered which will be called **real numbers.** These imaginary numbers are admitted as an outcome of the laws of algebra, and they will be supposed to enter into all operations in accordance with these laws. Thus, we say

$$\sqrt{-16} = \sqrt{16 \times -1} = \sqrt{16} \times \sqrt{-1} = \pm 4\sqrt{-1},$$

and the roots of the equation $x^2 - 2x + 5 = 0$ are

$$1 \pm 2\sqrt{-1}.$$

Also, it is only after admitting imaginary numbers that we can say that the equation treated has roots.

A number, as $1 + 2\sqrt{-1}$, in which one part is real and the other imaginary, is generally called a **complex number.**

It is readily seen that: *The roots of the equation* $ax^2 + bx + c = 0$ *will be*

(1) *real and unequal, if* $b^2 > 4ac$,

(2) *real and equal, if* $b^2 = 4ac$,

(3) *complex, if* $b^2 < 4ac$,

it being supposed that a, b, c, the given constants of the equation, are real.

EXERCISES

1. Resolve into linear factors :

$$x^2+7x-5\,; \quad 2x^2-7x+5\,;$$
$$3-5x-4x^2\,; \quad 7x^2-11x+5.$$

2. Solve the equations :

$$x^2-7x+3=0\,; \quad 3x^2-4x-5=0\,;$$
$$7-3x-5x^2=0\,; \quad 3x^2-11x+21=0.$$

3. Without solving the equations, determine the character of the roots—whether real or imaginary, and if real whether positive or negative—of the following :

$$x^2-9x+3=0\,; \quad 2x^2-7x-5=0\,;$$
$$2x^2+5x+1=0\,; \quad 3x^2-15x+10=0.$$

4. Construct the equation whose roots exceed those of the equation $x^2-7x+3=0$ by 1.

5. Construct the equation whose roots exceed those of the equation $ax^2+bx+c=0$ by h.

6. Construct the equation whose roots are twice those of the equation $x^2-5x+3=0$.

7. Construct the equation whose roots are m times those of the equation $x^2+px+q=0$.

8. Shew that each root of the equation $hx^2+2kx+h=0$ is the reciprocal of the other.

9. If m and n are the roots of the equation $2x^2-7x+3=0$ construct the equation whose roots are

$$(1)\ m^2 \text{ and } n^2\,; \quad (2)\ \frac{1}{m} \text{ and } \frac{1}{n}\,; \quad (3)\ m^2n \text{ and } mn^2.$$

10. If m and n are the roots of the equation $ax^2+bx+c=0$ construct the equation whose roots are

$$(1)\ m^2 \text{ and } n^2\,; \quad (2)\ \frac{1}{m} \text{ and } \frac{1}{n}\,; \quad (3)\ \frac{1}{m^2} \text{ and } \frac{1}{n^2}\,;$$

$$(4)\ m^3 \text{ and } n^3\,; \quad (5)\ m^2n \text{ and } mn^2\,; \quad (6)\ \frac{m}{n} \text{ and } \frac{n}{m}\,.$$

11. Find the condition that the equations $x^2+px+q=0$ and $x^2+rx+s=0$ may have a common root.

12. Find the conditions that the two equations

$$ax^2+bx+c=0\,; \quad px^2+qx+r=0$$

may have the same roots.

13. Shew that the value of the expression x^2-3x+2 is *zero* for $x=1$ or $x=2$, is *negative* for all values of x intermediate to 1 and 2, and is *positive* for all values of x exterior to the interval (1, 2).

14. Find for what values of x the expression x^2-x-12 is zero, for what values of x it is negative and for what values of x it is positive.

15. If the roots of the equation $x^2+px+q=0$ are known to be real, shew that the expression x^2+px+q is positive for values of x which are not roots and which do not lie between the roots.

If the roots are known to be equal, or if they are known to be imaginary, what can be inferred as to sign of the expression x^2+px+q?

16. Divide a given straight line of length a (units) so that the rectangle contained by the whole and one part may be equal to the square on the other part.

Interpret the roots in Euclid's construction.

6. **Equations of Degree Higher than the Second.** While the general equation of the third degree and that of the fourth degree admit solution it is not proposed to consider these solutions. However, there will be discussed certain examples of special types which do not require for their solution a knowledge of equations of degree higher than the second. The general equation of the fifth or of any higher degree does not admit algebraic solution.

It is to be remarked that the term degree is applied to an equation only after it has been brought to the standard form (if it is not already in that form) of a polynomial equated to zero. Thus, if an equation is fractional in the unknown it has to be cleared of fractions with respect to that unknown, or if it is irrational in the unknown it has to be brought to a rational form, before we can speak of its degree. It goes without saying that often the reduction can be anticipated and the degree discovered.

As in the case of the quadratic equation, when an equation of any degree is given in the form,

$$\text{polynomial in } x=0$$

it will be seen that *to a root of the equation corresponds a factor of the polynomial.* This fact is so important that a formal proof will be given. The following lemma will first be proved:

Lemma. *Let $f(x)$ be any polynomial in x; when $f(x)$ is divided by $x-m$ the remainder will be $f(m)$, i.e., the result of substituting m for x in the polynomial.*

Suppose $f(x)$ divided by $x-m$, in the ordinary way, the division terminating when a remainder not involving x is reached. The quotient will be a polynomial, which denote by $q(x)$, and the remainder, which denote by r, will not involve x. Then from the meaning of division we have

$$f(x)=q(x).\ (x-m)+r$$

an equality which holds for all values of x. Put, then, $x=m$ and it follows that

$$\begin{aligned} f(m)&=q(m).\ (m-m)+r \\ &=r. \end{aligned}$$

Thus r, the remainder, is the result of substituting m for x in the polynomial.

Theorem. *Let $f(x)$ be any polynomial in x; then, if m is a root of the equation $f(x)=0$, must $x-m$ be a factor of $f(x)$.*

Since m is a root of $f(x)=0$, then $f(m)=0$. Let, now, $f(x)$ be divided by $x-m$; as has been seen, the remainder is $f(m)$. But $f(m)$ is known to be zero; therefore, the remainder is zero, and consequently $x-m$ is a factor of $f(x)$.

The proof of the converse theorem is immediate.

Ex. 1. Solve the equation

$$x^4-8x^2+15=0.$$

This equation is of degree 4 in the unknown x, but if x^2 is regarded as the unknown the equation is quadratic. Then

$$(x^2)^2-8(x^2)+15=0;$$

which is satisfied if $x^2=3$ or if $x^2=5$,
i.e., if $x=\pm\sqrt{3}$, or if $x=\pm\sqrt{5}$.

Thus, the given equation of the *fourth* degree has the *four* roots $+\sqrt{3}$, $-\sqrt{3}$, $+\sqrt{5}$, $-\sqrt{5}$, and the polynomial x^4-8x^2+15 expressed as a product of linear factors is

$$(x-\sqrt{3})\ (x+\sqrt{3})\ (x-\sqrt{5})\ (x+\sqrt{5}).$$

Ex. 2. Solve the equation

$$x^3-1=0.$$

Plainly must

$$(x-1)(x^2+x+1)=0.$$

The equation will be satisfied by those values of x which will make $x-1$ and x^2+x+1 separately zero, *i.e.*, by the value 1 and the roots of

$$x^2+x+1=0,$$

which are found to be

$$\frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2}.$$

Thus, the given equation of the *third* degree has the *three* roots

$$1,\ \frac{-1+\sqrt{-3}}{2},\ \frac{-1-\sqrt{-3}}{2}.$$

The equation may be put in the form $x^3=1$ whence it appears that x is the cube root of unity, so that there are three such cube roots, one real, the other two imaginary. The two imaginary roots are the roots of the equation,

$$x^2+x+1=0.$$

Denote these roots by m and n. Then

$$m^3=1,\ n^3=1,\ m+n=-1,\ mn=1.$$

Multiply both sides of the last relation by m^2. Then

$$m^3n=m^2.$$

But $m^3=1$.

$$\therefore\quad n=m^2,$$

and consequently, *each imaginary root is equal to the square of the other.* Therefore, if one imaginary root be denoted by ω, the three cube roots of unity are

$$1,\ \omega,\ \omega^2.$$

The relation $mn=1$ declares that *each imaginary root is the reciprocal of the other.*

These relations should be verified by actual computation.

Ex. 3. Solve

$$12x^4+4x^3-41x^2+4x+12=0.$$

This equation of the fourth degree is of special form, the co-efficients of the terms equidistant from the beginning and the end of the arranged polynomial being equal. The method of solution of such equations is as follows :

Divide each side of the equation by x^2; this is permissible since x is a root of the equation and it is seen that $x=0$ is not a root. Then

$$12x^2+4x-41+4.\frac{1}{x}+12.\frac{1}{x^2}=0$$

$$\therefore \quad 12\left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)-41=0.$$

Now $x^2+\frac{1}{x^2}$ differs from the square of $x+\frac{1}{x}$ by 2; supply then 2 within the brackets of the first term, *i.e.*, virtually 2×12, and correct by subtracting 24. Then

$$12\left(x+\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)-65=0.$$

If $x+\frac{1}{x}$ be regarded as the unknown this last is a quadratic equation. By solving it is seen to be satisfied by

$$x+\frac{1}{x}=\frac{13}{6}, \text{ or } x+\frac{1}{x}=-\frac{5}{2}.$$

Take (I) $$x+\frac{1}{x}=\frac{13}{6}$$

$$\therefore \quad 6x^2-13x+6=0, \text{ whence } x=\frac{2}{3} \text{ or } \frac{3}{2}.$$

Take (II) $$x+\frac{1}{x}=-\frac{5}{2}$$

$$\therefore \quad 2x^2+5x+2=0, \text{ whence } x=-2 \text{ or } -\frac{1}{2}.$$

Therefore the equation has the four roots

$$\tfrac{2}{3},\ \tfrac{3}{2},\ -2,\ -\tfrac{1}{2}.$$

The equations I and II both being of the form

$$x+\frac{1}{x}=k$$

or, when reduced, of the form

$$x^2-kx+1=0$$

will have the product of their roots equal to 1, *i.e.*, each the reciprocal of the other. On this account an equation of the type proposed will have roots which fall into pairs of reciprocal numbers. Such an equation is called a **reciprocal equation.**

Ex. 4. Solve.

$$(x+1)(x+2)(x+3)(x+4)=1680.$$

If the first and the last factors on the left are multiplied, and also the mean factors the equation takes the form

$$(x^2+5x+4)(x^2+5x+6)=1680.$$

Now, treat x^2+5x+4 as the unknown, for the time being, and denote it by u. Therefore, the equation becomes

$$u(u+2)=1680$$

or

$$u^2+2u-1680=0$$

$$\therefore \quad u=40 \text{ or } -42.$$

Take (I)

$$x^2+5x+4=40$$

$$\therefore \quad x^2+5x-36=0$$

$$\therefore \quad x=-9 \text{ or } +4.$$

Take (II)

$$x^2+5x+4=-42$$

$$\therefore \quad x^2+5x+46=0$$

$$\therefore \quad x=\frac{-5\pm\sqrt{-159}}{2}.$$

The roots of the proposed equation of the *fourth* degree — *four* in number — are

$$4, \ -9, \ \frac{-5+\sqrt{-159}}{2}, \ \frac{-5-\sqrt{-159}}{2},$$

of which two are imaginary.

Ex. 5. Solve.

$$2x^2+6x-\sqrt{x(x+3)-1}=17.$$

It is readily seen that if this equation is rationalized it will be of the fourth degree. To avoid passing to an equation of this degree it is well to examine the forms appearing in the equation. By writing the equation thus:

$$2(x^2+3x)-\sqrt{x^2+3x-1}-17=0$$

it is seen that x occurs only in the grouping x^2+3x, or x^2+3x-1 since the radical signs requires that x^2+3x-1 be treated as a single quantity. We, therefore, write the equation thus:

$$2(x^2+3x-1)-\sqrt{x^2+3x-1}-15=0.$$

Denote $\sqrt{x^2+3x-1}$ by u.

$$\therefore\ 2u^2-u-15=0.$$

which is a quadratic in u.

$$\therefore\ u=3 \text{ or } -\tfrac{5}{2}.$$

Take (I) $u=3$.

$$\therefore\ \sqrt{x^2+3x-1}=3$$
$$\therefore\ x^2+3x-1=9$$
$$\therefore\ x^2+3x-10=0$$
$$\therefore\ x=2 \text{ or } -5.$$

Take (II) $u=-\tfrac{5}{2}$.

$$\therefore\ \sqrt{x^2+3x-1}=-\tfrac{5}{2}$$
$$\therefore\ x^2+3x-1=\tfrac{25}{4}$$
$$\therefore\ x^2+3x-\tfrac{29}{4}=0$$
$$\therefore\ \frac{-3\pm\sqrt{38}}{2}.$$

Thus, the original equation, virtually of the *fourth* degree, has the *four* roots

$$2,\ -5,\ \frac{-3+\sqrt{38}}{2},\ \frac{-3-\sqrt{38}}{2}$$

Of these roots, only the first two satisfy the equation if in evaluating $\sqrt{x^2+3x-1}$ we take only the positive root. The solution does not regard this limitation, and in its larger sense the equation should be said to have four roots.

EXERCISES

Solve the following equations : Do any two or three of these

1. $10x^4-29x^2+21=0$.
2. $35x^4-39x^3-4x^2-39x+35=0$.
3. $(x+3)(x+4)(x+5)(x+6)=5040$.
4. $(2x+1)(x-3)-\sqrt{x^2+x(x-5)+11}=16$.
5. $\dfrac{3x}{4+x^2}+\dfrac{4+x^2}{3x}=\dfrac{25}{12}$.
6. $12x^4-8x^3-15x^2-8x+12=0$. same as 2
7. $x^4-1=0$. factors
8. $(x^2+7x-5)(x+3)(x+4)=1050$. same as 3
9. $\sqrt[3]{x+20}-\sqrt[3]{x+1}=1$.
10. $(x+3)^2-\sqrt{(x+1)(x+5)+25}=89$.

7. Simultaneous Equations of the First Degree. In the expression $2x+3y-13$, the numbers 2, 3, -13 are constants and the numbers x, y are variables, and the expression is a function of x and y depending for its value on the values assigned to x and y.

Consider now the equation

$$2x+3y-13=0$$

or which is the same thing

$$2x+3y=13.$$

Plainly, it is an easy matter to find solutions of this equation, for if to y be assigned any value whatever, the equation determines a value of x which with that of y constitutes a solution. Thus, if $y=1$ it is found that $x=5$, and thus $x=5$, $y=1$ is a solution. The equation then imposes only a partial restraint upon the variables in the expression $2x+3y-13$, making the variation of one determine the variation in the other. The following table exhibits a series of solutions:

$x=6\frac{1}{2}$	5	$3\frac{1}{2}$	2	$\frac{1}{2}$	-1	.....
$y=0$	1	2	3	4	5	.....

Similar remarks apply to the equation

$$3x+2y-12=0$$

or

$$3x+2y=12,$$

and a series of solutions is given in the table,

$x=\ 4$	$3\frac{1}{3}$	$2\frac{2}{3}$	2	$1\frac{1}{3}$	$\frac{2}{3}$	.....
$y=\ 0$	1	2	3	4	5	.....

Among the solutions of the two equations we notice one, namely $x=2$, $y=3$, which is common, and we inquire whether it is the only common solution. In other words, we ask for all values of x and y that will satisfy the two equations

$$2x+3y=13 \qquad (1)$$
$$3x+2y=12 \qquad (2)$$

at the same time. Since we now suppose x and y to denote any such values, the x and the y in the two equations are the same. Multiplying both sides of (1) by 2 and both sides of (2) by 3 we have the equivalent set

$$4x+6y=26 \qquad (3)$$
$$9x+6y=36. \qquad (4)$$

Therefore, by subtraction

$$5x=10$$

or

$$x=\ 2.$$

Substituting this value of x in either equation we find the value of y to be 3, so that $x=2$, $y=3$ is the only solution common to the two equations. Thus, *the two linear equations in x and y serve completely to determine the values of x and y.*

In the same way the two general linear equations in two unknowns

$$ax+by=c,$$
$$a'x+b'y=c',$$

are seen to have one and only one solution, namely,

$$x=\frac{cb'-c'b}{ab'-a'b},\ y=\frac{ca'-c'a}{ba'-b'a}.$$

If it is a question of linear equations in three variables, a similar examination will shew that:

(1) *One linear equation allows any values whatever to be assigned to two of the variables, these two values determining that of the remaining variable.*

(2) *Two linear equations allow any value whatever to be assigned to one of the variables, this value determining those of the remaining variables.*

(3) *Three linear equations determine completely the values of the three variables.*

These statements are made on the supposition that the equations are *independent* and *consistent.* Thus, the equations

$$x+2y-\ z=3$$
$$2x+4y-2z=6$$

are not independent, the one following from the other, and should not be spoken of as two equations; also, the equations

$$x+2y-z=3$$
$$x+2y-z=7$$

cannot both be true, *i.e.*, are inconsistent, x, y, z, being supposed finite, and, therefore, cannot both aid in determining values of x, y, z.

EXERCISES

Solve the following systems of equations:

1. $5x+7y=50,$
   $4x+5y=37.$

✓ 2. $\frac{x}{3}-\frac{y}{6}=\frac{1}{2},$
   $\frac{x}{5}-\frac{3y}{10}=\frac{1}{2}.$

3. $\frac{x+y}{2}-\frac{x-y}{3}=16,$
   $\frac{x+y}{3}+\frac{x-y}{4}=22.$

✓ 4. $x+2y+3z=14,$
   $2x+3y+5z=23,$
   $3x+5y+7z=34.$

5. $x-y+z=11,$
   $2x-3y+9z=21,$
   $3x+4y-5z=17.$

8. **Simultaneous Equations Involving the Unknowns to a Degree Higher than the First.** In this Art. will be given the solutions of certain examples which illustrate classes of equations which may be solved without appeal to the solutions of equations of the third or the fourth degree.

Ex. 1. Solve,

$$\left.\begin{array}{r} 2x-3y=3 \\ x^2-xy+2y^2+x-4y=7 \end{array}\right\}$$

Here one of the given equations is linear and the other quadratic in the unknowns. Substitute in the quadratic for one of the unknowns its value, found from the linear equation, in terms of the other unknown.

From the linear equation

$$x=\frac{3y+3}{2}.$$

Therefore, by substitution in the quadratic equation,

$$\left(\frac{3y+3}{2}\right)^2-\left(\frac{3y+3}{2}\right)y+2y^2+\frac{3y+3}{2}-4y=7,$$

or

$$11y^2+2y-13=0,$$

a quadratic in y, as was easily foreseen.

$$\therefore\ y=1 \text{ or } -\tfrac{13}{11}.$$

If $y=1$, x, which equals $\frac{3y+3}{2}$, is found to be 3.

If $y=-\frac{13}{11}$, x which equals $\frac{3y+3}{2}$, is found to be $-\frac{3}{11}$.

The solutions then are

$$(x=3,\ y=1),\ (x=-\tfrac{3}{11},\ y=-\tfrac{13}{11}).$$

Any system of equations of the kind in question will, then, admit two solutions.

Ex. 2. Solve

$$\left.\begin{aligned}2x^2-4xy+3y^2&=6\\ x^2+\ xy-2y^2&=7\end{aligned}\right\}\quad(1)$$

The two equations are here quadratics. In general, two such equations can be solved only by solving a general equation of the fourth degree. The equations given are, however, of special form in that they contain no terms of one dimension.

Since the equations are presumably satisfied by some values of x and y, the value of y will be equal to the value of x multiplied by some number, not yet known. We may put, then,

$$y=mx\quad(2)$$

where m is a new unknown.

Substitute in the given equations : then

$$\left.\begin{aligned}2x^2-4mx^2+3m^2x^2&=6\\ x^2+mx-2m^2x^2&=7\end{aligned}\right\}\quad(3)$$

or

$$\left.\begin{aligned}x^2(2-4m+3m^2)&=6\\ x^2(1+m-2m^2)&=7\end{aligned}\right\}\quad(4).$$

Then, by the division of equal quantities,

$$\frac{2-4m+3m^2}{1+m-2m^2}=\frac{6}{7}.$$

Then, clearing of fractions, we have

$$33m^2-34m+8=0$$
$$\therefore \quad m=\tfrac{2}{3} \text{ or } \tfrac{4}{11}.$$

Take first $m=\frac{2}{3}$. Then substituting in the last of equations (4) we have

$$x^2(1+\tfrac{2}{3}-\tfrac{8}{9})=7$$
$$\text{or} \quad x=+3 \text{ or } -3$$

$$\left.\begin{array}{l} \text{If } x=+3,\ y=2 \\ \text{If } x=-3,\ y=-2 \end{array}\right\}, \text{ since } y=mx, \text{ and } m=\tfrac{2}{3}.$$

Take next, $m=\frac{4}{11}$. Then, as before,

$$x^2(1+\tfrac{4}{11}-\tfrac{32}{121})=7$$
$$\text{or} \quad x=+\frac{11}{\sqrt{19}} \text{ or } -\frac{11}{\sqrt{19}}.$$

$$\left.\begin{array}{l} \text{If } x=+\dfrac{11}{\sqrt{19}},\ y=\dfrac{4}{\sqrt{19}} \\ \text{If } x=-\dfrac{11}{\sqrt{19}},\ y=-\dfrac{4}{\sqrt{19}} \end{array}\right\}, \text{ since } y=mx, \text{ and } m=\tfrac{4}{11}.$$

Thus, the *two quadratic equations in two unknowns yield the four solutions*

$$\begin{array}{r|c|c|c|} x=+3 & -3 & +\dfrac{11}{\sqrt{19}} & -\dfrac{11}{\sqrt{19}} \\ \hdashline y=+2 & -2 & +\dfrac{4}{\sqrt{19}} & -\dfrac{4}{\sqrt{19}} \end{array}$$

Ex. 3. Solve

$$\left.\begin{array}{l} x^3-y^3=98 \\ x\ -y\ =\ 2 \end{array}\right\}.$$

We have here a system of one linear and one cubic equation. Since $x-y$ is a factor of x^3-y^3, and is not equal to zero, we have by division by equal values,

$$x^2+xy+y^2=49.$$

Substituting for x from $x-y=2$, we have

$$(2+y)^2+(2+y)y+y^2=49$$
$$\therefore \quad y^2+2y-15=0$$
$$\therefore \quad y=3, \text{ or } -5.$$
$$\text{If } y=3, \text{ then } x=5, \text{ since } x-y=2.$$
$$\text{If } y=5, \text{ then } x=-3, \text{ since } x-y=2.$$

Therefore, two solutions are yielded, namely,

$$(x=5,\ y=3),\ (x=-3,\ y=-5).$$

These solutions would also have been obtained if we had substituted for x, from $x-y=2$, in the original cubic.

EXERCISES

Solve the following systems of equations:

1. $x+y=7,$
$x^2+3xy-5y^2+2x-y=12.$

2. $x+y=8,$
$x^2+y^2=34.$

3. $x^2+xy=30,$
$6x^2-y^2=5.$

4. $2x^2-3xy+5y^2=7,$
$x^2+5xy-11y^2=3.$

5. $x^3+y^3=189,$ Divide
$x+y=9.$

6. $x^4+x^2y^2+y^4=21,$ [illegible]
$x^2+xy+y^2=7.$

7. $3x+4y=18,$
$2x^2-7xy-y^2+5x+11y=0.$

8. $xy=42,$
$x^2+y^2=85.$

9. $4x+6y=xy,$
$12x+9y=2xy.$

10. $2(x^2+y^2)-3(x+y)=5,$
$4xy=15.$

11. $3x+5y=4x^2-6y^2,$
$2x+12y=x^2+y^2.$

[illegible] any on [illegible]

EXAMPLES

1. Plot the curve (*i.e.*, construct the graph) of the function

$$x^3-2x^2-5x+5$$

for values of x between -3 and 4, and state any inferences as to the nature and value of the roots of the equation

$$x^3-2x^2-5x+5=0.$$

(NOTE.—Denote the function by y in place of u as earlier. The axes will be X′OX, YOY′, the former the x-axis and the latter the y-axis for measurements determining the curve.)

2. The population of a city at certain stated dates is given by the table:

Date	1850	1860	1870	1880	1890	1900
Population	37,000	43,600	52,700	61,400	66,300	69,800

Plot a curve shewing the change in population.

3. The quoted prices of a certain railway stock at intervals of 15 days were: 124, 119, 111, 100, 96, 94. Plot a curve exhibiting the change in value of the stock.

4. The volume of a gas under certain pressures is given by the following table, the pressure being in pounds to the square inch and the volume in cubic feet:

Pressure	12	15	18	24	27
Volume	90	72	60	45	40

Plot a curve shewing the variation in volume under the changing pressure and conjecture the volume under pressure 21.

5. Noting that the equation

$$2x + 3y = 24$$

gives y as a function of x, namely

$$y = \frac{24 - 2x}{3},$$

construct the graph of the function.

(Note.—This graph, seen to be a straight line, is also spoken of as the graph of the *equation* $2x + 3y = 24$.)

6. Construct, referring them to the same axes, the graphs of each of the equations,

$$3x + 4y = 24,$$
$$2x + 3y = 17.$$

Also, regarding these as two simultaneous equations, infer from the graph the solution.

7. Plot the curve of the function

$$x^3 - 5x^2 + 11x - 14$$

between $x = 0$ and $x = 4$, and obtain an approximate value of a root of the equation

$$x^3 - 5x^2 + 11x - 14 = 0.$$

8. Find the values of m if the equation

$$(m - 1)\, x^2 - (4m + 4)\, x + (7m + 1) = 0$$

is known to have equal roots.

9. One root of the equation

$$x^2 - (k + 1)\, x + (2k + 1) = 0$$

is known to exceed the other by 2. Find the values of k and find the roots that meet this condition.

10. The difference between the roots of the equation $ax^2 + bx + c = 0$ is equal to the difference between the roots of the equation $px^2 + qx + r = 0$, shew that

$$p^2\, (b^2 - 4ac) = a^2\, (q^2 - 4pr).$$

11. Solve

$$(x + 3)\, (x + 5)\, (x + 7)\, (x + 9) = 3465.$$

12. Solve

$$2x^2 + 6x - 41 = \sqrt{x^2 + 3x + 7}.$$

13. Solve

$$9x^4 - 51x^3 + 88x^2 - 51x + 9 = 0.$$

14. Solve

$$x^5 = 1.$$

15. Solve

$$x^6 = 1.$$

16. Solve

$$\begin{aligned} 3x^2 - 5xy - 7x + 11y + 14 &= 0, \\ 4x - y - 7 &= 0. \end{aligned}$$

17. Solve

$$\begin{aligned} xy + x + y &= 11, \\ xy\, (x + y) &= 30. \end{aligned}$$

18. Solve

$$\begin{aligned} x^2 + 2xy &= 16, \\ xy + y^2 &= 15. \end{aligned}$$

19. Solve

$$x^2 - 2y^2 = 4y,$$
$$3x^2 + xy - 2y^2 = 16y.$$

20. Solve

$$x + y = 4,$$
$$x^4 + y^4 = 82.$$

21. Shew that an integral quadratic function can be resolved into linear factors in only one way.

Would $(2x-3)(x-2)$, $(x-\frac{3}{2})(2x-4)$, $2(x-\frac{3}{2})(x-2)$ be regarded as different factor-expressions of $2x^2 - 7x + 6$?

22. The expression $ax^2 + bx - 11$ is known to be equal to -7 if $x = 4$, and to be equal to 4 if $x = 5$. Find its value if $x = 3$.

23. Find the price of eggs a dozen when 15 more in a dollar's worth would mean a reduction of 4 cents on the price.

24. Two travellers A and B set out at the same time from two places P and Q respectively, and travel so as to meet. When they meet it is found that A has travelled 30 miles more than B, and that A will reach Q in 4 days, and B will reach P in 9 days, after they meet. Find the distance between P and Q.

25. Divide a line 40 inches long into two parts such that the hypotenuse of the right-angled triangle of which the two parts are the sides shall be the least possible.

26. An article is sold at a loss of as much per cent. as it is worth in dollars. Shew that it cannot be sold for more than 25 dollars.

27. The base of a semi-circle whose radius is a is divided into two parts and on these parts as bases semi-circles are described. How must the base be divided if the area bounded by the three circles is to be a maximum?

Begin here

CHAPTER II

RATIO AND PROPORTION

1. **Preliminary.** When two different numbers, as for example 2 and 3, are given, the fact of difference may be regarded in different ways. Thus, we may say that the absolute difference is 1, meaning that 2 is less than 3 by 1, or that 3 is greater than 2 by 1; or we may consider the relative values of the two numbers and say that 2 is two thirds of 3 or that 3 is three halves of 2. In regarding two numbers in this latter way we arrive at the concept of **ratio,** and we see that the ratio of 2 to 3 (in symbols 2 : 3) is expressed by the fraction $\frac{2}{3}$.

The ratio of two numbers being expressed by a fraction, we may speak of the value of a ratio and may study the properties of ratios in the fractions by which they are represented. Indeed, all the theorems proved with respect to fractions are theorems in ratios.

In the ratio 2 : 3, the numbers 2 and 3 are called the terms of the ratio, the former the antecedent, and the latter the consequent.

If two numbers are equal we still speak of their ratio; this ratio is expressed by the number 1 and is called a **ratio of equality.** A ratio of two positive numbers in which the antecedent exceeds the consequent is called a **ratio of greater inequality;** one, in which the antecedent is less than the consequent, a **ratio of less inequality.**

One ratio is equal to, greater than, or less than another according as the fraction which represents that ratio is equal to, greater than, or less than the fraction representing the other ratio. When two ratios are equal the four numbers in order are said to be in **proportion.** Thus, 2, 3, 10, 15 are in proportion since

$$\frac{2}{3}=\frac{10}{15}.$$

The statement of the proportion in symbols is

$$2 : 3 :: 10 : 15$$

which is read, *two is to three as ten is to fifteen.* To find whether or not four given numbers are in proportion it is necessary only to examine whether or not two fractions are equal.*

When several numbers in sequence as $a, b, c, d, \ldots\ldots$ are such that $a:b::b:c::c:d::\ldots\ldots$, these numbers are said to be in **continued proportion.**

If three numbers $a, b, c,$ are in continued proportion b is said to be a **mean proportional** between a and c, and c is said to be a **third proportional** to a and b.

If there are several (say three) ratios as $a:b$, $c:d$, $e:f$, the ratio $ace:bdf$ is said to be **compounded** of the given ratios, and hence if there are several (say four) numbers in sequence $p, q, r, s,$ then the ratio $p:s$ is said to be compounded of the ratios $p:q$, $q:r$, $r:s$.

The ratios $a^2:b^2$, $a^3:b^3$, ... are called the **duplicate**, the **triplicate** ..... of the ratio $a:b$.

2. The following propositions are of importance:

(I). *A ratio of less inequality is increased, a ratio of greater inequality is diminished, and a ratio of equality is unchanged by the addition of the same positive number to each term.*

Let the ratio be $a:b$ where a and b are positive, and let the positive number k be added to each term so that the resulting ratio is $a+k:b+k$. The two ratios are expressed by the fractions

$$\frac{a}{b},\ \frac{a+k}{b+k}.$$

Then

$$\frac{a+k}{b+k}-\frac{a}{b}=\frac{b(a+k)-a(b+k)}{b(b+k)}$$

$$=\frac{(b-a)k}{b(b+k)}.$$

Now, $k, b, b+k$ are positive so that the sign of this last fraction is that of $b-a$. Therefore the difference $\frac{a+k}{b+k}-\frac{a}{b}$ is positive, negative or zero according as b is greater than, less than, or equal to, a. Thus, the ratio $a:b$ has been increased if b is greater than a, *i.e.*, if the ratio is one of less inequality; diminished, if b is less than a, *i.e.*, if the ratio is one of greater inequality; and unchanged if b is equal to a, *i.e.*, if the ratio is one of equality.

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* In contrast with the definition and treatment of proportion here given, Euclid's definition and treatment appear difficult and forced. This is due to the fact that Euclid provides for cases not here contemplated, namely, those in which appear irrational numbers *of whatever kind.* To establish a theory of operations of such numbers is certainly not easier than to comprehend the meaning and feel the beauty of his definition.

(II). *If $a:b$ and $c:d$ are two unequal ratios of positive quantities, the ratio $a+c:b+d$ is intermediate in value to these two ratios.*

The given ratios are expressed by

$$\frac{a}{b}, \frac{c}{d}.$$

Let $\frac{a}{b}$ be the greater of these fractions and denote its value by k. Then

$$\frac{a}{b}=k$$

$$\therefore \quad a=bk$$

Also
$$\frac{c}{d}<k$$

$$\therefore \quad c<dk$$

$$\therefore \quad a+c<bk+dk,$$

$$\text{i.e.,} \quad a+c<(b+d)k$$

$$\therefore \quad \frac{a+c}{b+d}<k; \text{ i.e., } \frac{a+c}{b+d}<\frac{a}{b}.$$

In like manner
$$\frac{a+c}{b+d}>\frac{c}{d}.$$

Therefore, the ratio $a+c:b+d$ lies in value between $a:b$ and $c:d$.

This theorem can easily be generalized.

(III). *If four quantities are proportionals, the product of the extremes is equal to the product of the means.*

Let $a:b::c:d$; it is required to prove that $ad=bc$. It is given that

$$\frac{a}{b}=\frac{c}{d}.$$

Multiply each side of this equality by bd. Then

$$ad=bc.$$

Conversely if $ad=bc$ then shall $a:b::c:d$.

For if

$$ad = bc$$

then, by dividing each side by bd, we obtain

$$\frac{a}{b} = \frac{c}{d},$$

i.e., $$a : b :: c : d.$$

Cor. If $ad = bc$, *it is plain also that*

$$\frac{a}{c} = \frac{b}{d};$$

therefore, if $a : b :: c : d$ *it follows that* $a : c :: b : d$.

(IV). *If* $a : b :: c : d$ *then each ratio is equal to the ratio expressed by*

$$\frac{la + mc}{lb + md}.$$

Since $\frac{a}{b}$ and $\frac{c}{d}$ are equal, we may denote their common value by v.

Then $a = bv$, $c = dv$, and

$$\frac{la + mc}{lb + md} = \frac{lbv + mdv}{lb + md}$$

$$= \frac{v\,(lb + md)}{lb + md}$$

$$= v, \text{ the common value of } \frac{a}{b}, \frac{c}{d}.$$

The theorem may be shewn to be true without introducing the quantity v.

For if

$$\frac{a}{b} = \frac{c}{d}$$

then $$\frac{a}{c} = \frac{b}{d}$$

$$\therefore \frac{la}{c} = \frac{lb}{d}$$

$$\therefore \frac{la}{c} + m = \frac{lb}{d} + m$$

$$i.e., \frac{la + mc}{c} = \frac{lb + md}{d}$$

$$\therefore \frac{la + mc}{lb + md} = \frac{c}{d}\left(= \frac{a}{b}\right).$$

The theorem may be stated thus:

If two fractions are equal, each is equal to the quotient of the sum (or difference) of any multiples of their numerators by the sum (or difference) of the same multiples of corresponding denominators.

(V). *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{la+mb}{pa+qb}=\frac{lc+md}{pc+qd}$,

Since $\frac{a}{b}=\frac{c}{d}$ it follows that $\frac{a}{c}=\frac{b}{d}$. By theorem (IV) each of these latter is equal to $\frac{la+mb}{lc+md}$, as also to $\frac{pa+qb}{pc+qd}$.

$$\therefore \frac{la+mb}{lc+md}=\frac{pa+qb}{pc+qd}$$

$$\therefore \frac{la+mb}{pa+qb}=\frac{lc+md}{pc+qd}.$$

The theorem may be given verbal statement.

3. **Illustrative Examples.** The following examples may repay study:

Ex. 1. If $ax-by=0$, find the ratio of x to y.

Since

$$ax-by=0,$$

it follows that

$$ax=by.$$

Therefore, dividing each side by ay, we have

$$\frac{x}{y}=\frac{b}{a}$$

i.e., $x:y::b:a.$

Thus: A homogeneous linear equation in x and y, *i.e.*, one which, when brought to the form in which zero is the right number, has every term of one dimension in x and y, while allowing x and y each to have different values, determines the ratio of x to y.

Ex. 2. If

$$ax+by+cz=0$$

and

$$px+qy+rz=0$$

find the ratios $x:y:z$.

Here we have two homogeneous linear equations in x, y, z. As has been seen two such equations do not determine x, y, z, and x, y, z may vary while satisfying the equations.

Multiplying each member of the first equation by r and each member of the second by c, we have,

$$\begin{aligned} arx+bry+crz&=0\\ cpx+cqy+crz&=0 \end{aligned}$$

Therefore, eliminating z, we have

$$x(cp-ar)-y(br-cq)=0$$

$$\therefore \quad x(cp-ar)=y(br-cq)$$

$$\therefore \quad \frac{x}{y}=\frac{br-cq}{cp-ar}$$

$$\text{or} \quad \frac{x}{br-cq}=\frac{y}{cp-ar}.$$

By eliminating y we have in like manner

$$\frac{x}{br-cq}=\frac{z}{aq-bp}.$$

Therefore, the relations

$$\frac{x}{br-cq}=\frac{y}{cp-ar}=\frac{z}{aq-bp}$$

give the ratios $x:y:z$.

Thus, while x, y, z may vary, the variation is such as to keep the *ratios* $x:y:z$ constant.

The expressions for these ratios are important, and it is easy to devise a method of writing them down from the three triads

$$\left\{\begin{matrix} x & y & z\\ a & b & c\\ p & q & r \end{matrix}\right\}.$$

Ex. 3. Solve

$$\begin{aligned} 2x-3y+z&=0,\\ x+2y-2z&=0,\\ x^2+2y^2-z^2&=68. \end{aligned}$$

From the first two equations

$$\frac{x}{6-2}=\frac{y}{1+4}=\frac{z}{4+3}$$

$$i.e., \quad \frac{x}{4}=\frac{y}{5}=\frac{z}{7}$$

Denote the common value of these ratios by k.

$$\therefore \quad x=4k,\ y=5k,\ z=7k.$$

Substitute in the third of the given equations.

$$\therefore\ 16k^2+50k^2-49k^2=68$$

$$\therefore\ k^2=4$$

$$\therefore\ k=+2 \text{ or } -2$$

$\therefore$ the solutions are

(1) $(x=8,\ y=10,\ z=14)$,
(2) $(x=-8,\ y=-10,\ z=-14)$.

EXAMPLES

1. If $12x^2-41xy+35y^2=0$ find the ratio of x to y.

2. For what value of x will $5+x$, $7+x$, $11+x$ be in continued proportion?

3. Find the number which added to each term of the ratio 5 : 8 will yield the ratio 4 : 5.

4. The lengths of two rectangles are in the ratio $a:a'$; their breadths are in the ratio $b:b'$. Find the ratio of their areas.

5. Find the mean proportional between 12 and 75.

6. If $\frac{a}{b}=\frac{c}{d}$, then will

(1) $$\frac{a^2b+ab^2}{a^3-b^3}=\frac{c^2d+cd^2}{c^3-d^3};$$

(2) $$\frac{3a^{10}+7b^{10}}{a^5b^5}=\frac{3c^{10}+7d^{10}}{c^5d^5}.$$

Shew that these are illustrations of the theorem: *If* $\frac{a}{b}=\frac{c}{d}$ *then any fraction formed with numerator and denominator homogeneous in a and b will be equal to the fraction similarly formed of c and d.*

7. If $\frac{a}{b}=\frac{c}{d}$, then

(1) $$\frac{ma^2+nc^2}{mb^2+nd^2}=\frac{ac}{bd};$$

(2) $$\frac{3a^3+5c^3}{3b^3+5d^3}=\frac{4a^2c+7ac^2}{4b^2d+7bd^2}.$$

8. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then

(1) $\frac{la^2+mc^2+ne^2}{lb^2+md^2+nf^2}=\frac{pce+qea+rac}{pdf+qfb+rbd}$;

(2) $\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf}$;

(3) $\frac{(la+mc+ne)^3}{(lb+md+nf)^3}=\frac{(c+e)(e+a)(a+c)}{(d+f)(f+b)(b+d)}$;

Shew that these are illustrations of the theorem: "*If two or more fractions are equal, then any fraction whose numerator is a homogeneous expression in the numerators, and denominator an expression similarly formed of the denominators, is equal to any other fraction whose numerator is homogeneous and of the same degree in the numerators, and denominator similarly formed of the denominators.*"

9. If

$$\frac{l}{b-c}=\frac{m}{c-a}=\frac{n}{a-b},$$

then

$$l+m+n=0$$

10. If $$\frac{x+y}{x-y}=\frac{y+z}{5(y-z)}=\frac{z+x}{7(z-x)}$$

then

$$20x+21y+6z=0$$

and of the quantities x, y, z, supposed real, two are of one sign and the remaining one of opposite sign.

11. If $$\frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}$$

then $x=y=z$.

12. If $$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$$

then

$$(b-c)\,x+(c-a)\,y+(a-b)\,z=0.$$

13. If

$$\frac{a}{x^2-yz}=\frac{b}{y^2-zx}=\frac{c}{z^2-xy}$$

then $$ax+by+cz=(a+b+c)\,(x+y+z).$$

14. If
$$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$$
then
$$(a+b+c)(yz+zx+xy)=(x+y+z)(ax+by+cz).$$

15. If
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c} \text{ then}$$
$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

16. If
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c},$$
then
$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(x+y+z)^2+(a+b+c)^2}{(x+y+z)+(a+b+c)}.$$

17. Solve
$$x-5y+2z=0.$$
$$7x+9y-6z=0,$$
$$yz-2zx+xy=4.$$

18. Solve
$$7x-11y+z=0,$$
$$13x+5y-49z=0,$$
$$3x^2-5y^2-z^2=x+2y-z.$$

19. Solve
$$yz-3zx+2xy=0,$$
$$5yz+6zx-18xy=0,$$
$$4x-5y+7z=21.$$

20. Shew that the three equations
$$x+y-z=0,$$
$$7x-9y+3z=0,$$
$$2x+5y-3z=0,$$
cannot all be true.

21. If $\frac{x}{a} = \frac{y}{b}$ then will $ax + by$ be a mean proportional between $x^2 + y^2$ and $a^2 + b^2$, and conversely.

22. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ then will

$$(a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) = (ax + by + cz)^2$$

and conversely.

23. If $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$ are three ratios, not all equal, of positive quantities, and if l, m, n are positive, then the ratio

$$\frac{lx + my + nz}{la + mb + nc}$$

lies in value between the greatest and the least of the given ratios.

CHAPTER III

VARIATION

1. **Preliminary.** In the applications of algebra there frequently present themselves quantities which are undergoing or may be supposed to undergo change, so that the numbers which measure them are variables. Any problem concerned with such variable quantities will involve two or more variable numbers, and these numbers are related so that a change in any one will cause at least one other number to change. That part of the subject which has to do with such related variables and the laws of their dependence on one another is called **variation.**

The following illustrations should be examined in order that the formal theorems may be more readily understood.

Illustration 1. An observer on a railway train notes that at a certain instant he is passing a mile-post; at the end of $1\frac{1}{2}$ minutes he notes that he passes the next mile-post, at the end of 3 minutes the next, at the end of $4\frac{1}{2}$ minutes the next, at the end of 6 minutes the next, and at the end of $7\frac{1}{2}$ minutes the next. What inference is to be drawn as to the motion of the train?

Let s (miles) be the distance travelled by the train in t (minutes) from the time when the observation began. As time passes t changes and s varies *with* t, or, in other words, a change in t necessitates a change in s.

Now let us take note of the observed facts. When t changes from $1\frac{1}{2}$ to 3, s changes from 1 to 2, and, as $\frac{1\frac{1}{2}}{3}=\frac{1}{2}$, the change in s is proportionate to the change in t; next when t changes from 3 to $4\frac{1}{2}$, s changes from 2 to 3, and as $\frac{3}{4\frac{1}{2}}=\frac{2}{3}$ the change in s is proportionate to the change in t, and so for the other given values of s and t. It thus appears that the change in s is proportionate to the change in t, and we infer that the train is moving uniformly.

When two variable quantities are so related that any change in the one implies a proportionate change in the other, then each is said to **vary as** the other.

Here s varies as t (or t varies as s), and this is written

$$s \propto t.$$

Illustration 2. A small bullet is allowed to fall freely from a height and at the end of 1 sec., 2 sec., 3 sec. it is found to have fallen through 16·1 ft., 64·4 ft., 144·9 ft. What seems to be the relation between the time measured from the instant the bullet was let fall and the distance through which it has fallen in that time?

Let t and s measure the time and distance in question. It is at once plain that s while *varying with* t does not vary as t. As t changes from 1 to 2, s changes from 16·1 to 64·4; now, $\frac{16\cdot 1}{64\cdot 4}=\frac{1}{4}=\frac{1^2}{2^2}$, so that the change in s is proportionate to the change in the square of t. So when t changes from 2 to 3, s changes from 64·4 to 144·9, and, as $\frac{64\cdot 4}{144\cdot 9}=\frac{4}{9}=\frac{2^2}{3^2}$, the change in s is proportionate to the change in the square of t. Hence, from the observations given we are led to suppose that s varies as the square of t. Investigations shew that this is the law of bodies falling freely. We say then that s varies as t^2, and write

$$s \propto t^2.$$

Illustration 3. The volume of a certain gas under pressure 14 (pounds to the square inch) is 108 (cubic feet); under pressure 21 the volume is found to be 72, and under pressure 28 the volume is found to be 54. In what way do the volume and the pressure seem to be related?

Let p and v denote the measures of the pressure and the volume.

When p changes from 14 to 21, v changes from 108 to 72; now $14:21$ as $2:3$, and $108:72$ as $3:2$ or as $\frac{1}{2}:\frac{1}{3}$ so that the change in the volume is proportionate to the change in the reciprocal or the inverse of the pressure. The same is found for the change from pressure 21 to pressure 28 so that when the pressures are as $2:3:4$ the volumes are as $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$. Here v is said to **vary inversely** as p and we write

$$v \propto \frac{1}{p}.$$

2. **Theorem.** The following theorem is the one fundamental proposition for the case in which appear only two variables of which one may be called the dependent, the other the independent variable.

If $y \propto x$, and if x be allowed to vary, then the value of y corresponding to any value taken by x is equal to the product of that value of x and some constant number, i.e., a number which, as x and therefore y change, does not change.

Here x and y are the measures of quantities so that we may speak of the ratio $x:y$ while it might not be permissible to speak of the ratio of the quantities measured by them.

Let x take *any* values $x_1, x_2, x_3, x_4, \ldots\ldots$ and let the corresponding values of y be $y_1, y_2, y_3, y_4, \ldots\ldots$

Then since $y \propto x$

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}$$

$$\therefore \frac{y_1}{x_1} = \frac{y_2}{x_2}$$

Similarly $$\frac{y_1}{x_1} = \frac{y_3}{x_3} = \frac{y_4}{x_4} = \ldots\ldots$$

$\therefore$ since $$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \frac{y_4}{x_4} = \ldots\ldots$$

it follows that, as x and therefore y change, the ratio of y to x does not change, *i.e.*,

$$\frac{y}{x} = k \text{ (a constant)}$$

$$\therefore y = kx$$

as it was required to prove.

Conversely: *If y and x are two related variables and if, as x and therefore y change, $y = kx$ where k is a constant, then $y \propto x$.*

For let x_1, x_2 be *any* two values of x, and y_1, y_2 the corresponding values of y. Then by the condition given

$$y_1 = kx_1,$$
$$y_2 = kx_2$$

$$\therefore \text{ by division} \qquad \frac{y_1}{y_2} = \frac{x_1}{x_2}.$$

Thus, *any* change in the value of x requires a proportionate change in the value of y, *i.e.*, $y \propto x$.

Ex. 1. The velocity of a particle falling freely from rest varies as the time from rest. At the end of $1\frac{1}{2}$ seconds the velocity is observed to be 48·3 feet a second. Find the relation giving the velocity in terms of the time, and the velocity at the end of 2 seconds.

Let v (feet a second) be the velocity at the end of time t (seconds) from rest. Then by the given condition

$$v \propto t$$
$$\therefore \; v = kt$$

where k is a constant whose value is to be found.

If $t = 1\frac{1}{2}$, v is 48·3.

$$\therefore \; 48{\cdot}3 = k \times 1\tfrac{1}{2}$$
$$\therefore \qquad k = 32{\cdot}2.$$

The value of k having been found we may write

$$v = 32{\cdot}2t$$

as the relation giving v in terms of t.

If $t = 2$, $v = 32{\cdot}2 \times 2 = 64{\cdot}4$, and the velocity at the end of 2 seconds is 64·4 feet a second.

Ex. 2. The space through which a particle falls freely from rest is known to vary as the square of the time from rest. In 1 second a particle is observed to fall 16·1 feet. Find the relation connecting the space and the time, the space through which the particle would fall in 3 seconds, and the space through which it would fall in the third second.

Let t and s measure the time in seconds and the space in feet from rest. Then

$$s \propto t^2$$
$$\therefore \; s = kt^2$$

where k is a constant whose value is not yet known.

If $t=1$, $s=16{\cdot}1$.

$$\therefore\ 16{\cdot}1=k\times1^2$$

$$\therefore\ k=16{\cdot}1.$$

Thus the relation connecting s and t is

$$s=16{\cdot}1t^2.$$

If $t=3$, $s=16{\cdot}1\times3^2=144{\cdot}9$, and the space through which the particle falls in 3 seconds is 144·9 feet.

Similarly, in 2 seconds the particle falls through $16{\cdot}1\times2^2$ or 64·4 feet. Therefore in the third second the particle falls through $144{\cdot}9-64{\cdot}4$ or 80·5 feet.

Ex. 3. The volume of a gas under change of pressure varies inversely as the pressure. The volume of a certain quantity of gas under pressure 15 (pounds on the square inch) is 96 (cubic feet). Find the volume under a pressure of 18.

Let v and p measure the volume and the pressure in the units indicated. Then

$$v\propto\frac{1}{p}$$

$$\therefore\ v=k.\frac{1}{p}$$

where k is a constant whose value is not yet known.

If $p=15$, $v=96$.

$$\therefore\ 96=k.\frac{1}{15}$$

$$\therefore\ k=15\times96$$

$$\therefore\ v=15\times96.\frac{1}{p}.$$

Therefore, if $p=18$, $v=15\times96\times\frac{1}{18}=80$ and the volume of the gas under pressure 18 is 80 cubic feet.

EXERCISES

1. The area of a circle is known from geometry to vary as the square of the radius. The area of a circle of radius $3\frac{1}{2}$ is found to measure 38·5. Find the formula for the area of a circle.

2. The surface of a sphere is known to vary as the square of the radius. A sphere of radius $1\frac{3}{4}$ is found to have a surface area of 38·5. Find the formula for the area of the surface of a sphere.

3. The volume of a sphere varies as the cube of its radius. Metal spheres of radii 3, 4, 5 are melted and cast into a single sphere. Find its radius.

4. The *illumination* from a source of light varies inversely as the square of the distance. At what distance from the source must a small sheet of paper be placed to receive $3\frac{3}{8}$ times as much light as when at a distance 15 feet from the source? (Why a *small* sheet?)

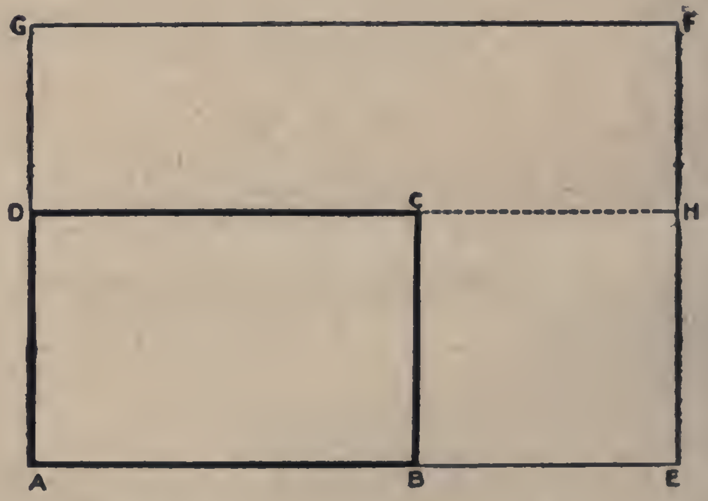
5. The square of the time of a planet's revolution about the sun varies as the cube of its mean distance from the sun. The mean distances of Venus and the Earth being given as in the ratio of 18 : 25 find in days the time of revolution of Venus.

6. If x and y vary in such a way that their product is constant, then y varies inversely as x or x varies inversely as y.

3. Problems Involving more than Two Variables.

Up to this point only two variables have appeared in any problem. In the following example appear three variables:

Ex. From geometry it is known that the area of a rectangle depends only on its base and its altitude; it is known also that the area varies as the base when the altitude is constant and as the altitude when the base is constant. It is required to study the variation in the area when both the base and the altitude vary.



Let A, b, h measure the area, the base and the height of a rectangle ABCD. Let b change to *any* value $b'(=\text{AE})$ and h to *any* value $h'(=\text{AG})$, and *as a result* suppose A to change to A′. We seek a relation between the old values A, b, h, and the new values A′, b', h'.

First, suppose b to change to b', h not changing, so that the rectangle ABCD becomes the rectangle AEHD, and let the area of AEHD be A_1. In this change the base has changed, the altitude not changing, and therefore by what is given

$$\frac{A}{A_1}=\frac{b}{b'} \qquad \text{(I)}.$$

Next, suppose h to change to h', b' not changing, so that the rectangle AEHD becomes the rectangle ABCD. In this change the altitude has changed, the base not changing, and therefore by what is given

$$\frac{A_1}{A'}=\frac{h}{h'} \qquad \text{(II)}.$$

Therefore from (I) and (II), by multiplication,

$$\frac{A}{A_1}\cdot\frac{A_1}{A'}=\frac{b}{b'}\cdot\frac{h}{h'}$$

or

$$\frac{A}{A'}=\frac{bh}{b'h'}.$$

Thus, if b and h both change in any way the change in the *product* bh is proportionate to the change in A and therefore A varies as bh, or the area varies as the product of (the measures of) the base and the altitude.

The preceding is a simple illustration of the following theorem, the proof of which may now be briefly stated.

Theorem. *If x is variable depending for its value on the two variables y and z, which are independent of each other, and if $x \propto y$ when z is constant, and $x \propto z$ when y is constant, then $x \propto yz$ when y and z both vary.*

Let y change to any value y' and z to any value z' and as a result suppose x to change to x'. First suppose y to change to y', z not changing, which is possible since y and z are independent, and suppose that *as a result* x changes to x_1.

Then, (x, y, z), (x_1, y', z), (x', y', z') are three sets of corresponding values of the variables.

In passing from the first to the second, y changes, z remaining constant.

$$\therefore \quad \frac{x}{x_1}=\frac{y}{y'} \qquad \text{(I)}.$$

In passing from the second to the third, z changes, y remaining constant.

$$\therefore \quad \frac{x_1}{x'}=\frac{z}{z'} \qquad \text{(II)}.$$

Then from (I) and (II), by multiplication,

$$\frac{x}{x_1}\cdot\frac{x_1}{x'}=\frac{y}{y'}\cdot\frac{z}{z'}$$

or

$$\frac{x}{x'}=\frac{yz}{y'z'}.$$

Thus the change in x is proportional to the change in the product yz, *i.e.*, $x \propto yz$.

When x varies as the product of two or more variable quantities it is said to vary **jointly** as those quantities.

Further complexities in variation will be developed in the exercises and examples which follow.

Ex. 1. The area of a rectangle is known from geometry to vary as the base when the altitude is constant and as the altitude when the base is constant. The area of a square of side 1 being the unit of area find the formula for the area of a rectangle.

Let A, b, h measure the area, the base and the altitude of the rectangle. Then $A \propto b$ when h is constant and $A \propto h$ when b is constant and b and h are independent.

$$\therefore \quad A \propto bh$$

$$\therefore \quad A = k.bh,$$

where k is a constant, *i.e.*, the same for all rectangles.

If $b=1$ and $h=1$, then $A=1$

$$\therefore \quad 1=k.1\times 1$$

$$\therefore \quad k=1$$

$$\therefore \quad A=bh,$$

the formula required.

Ex. 2. It is shewn in works on geometry that the volume of a cone varies as the square of the radius of the base when the altitude is constant and as the altitude when the base is constant. A cone of height 1 and with a base of radius 1 is measured and found to be $\frac{22}{21}$. Find the formula for the volume of a cone.

Let v, h, r measure the volume, the altitude and the radius of the base of the cone. Then

$$v \propto r^2 \text{ when } h \text{ is constant}$$

also

$$v \propto h \text{ when } r^2 \text{ (or } r\text{) is constant}$$

$$\therefore\ v \propto hr^2$$

$$\therefore\ v = k.hr^2,$$

where k is constant.

If $r=1$ and $h=1$, we are given that $v=\frac{22}{21}$

$$\therefore\ \tfrac{22}{21} = k.1 \times 1^2$$

$$\therefore\ k = \tfrac{22}{21}$$

$$\therefore\ v = \tfrac{22}{21}\,hr^2,$$

the formula required.

Ex. 3. The volume of a gas depends upon the temperature and the pressure. In works on physics it is given that the volume varies as the absolute temperature when the pressure is kept constant and inversely as the pressure when the temperature is kept constant. The volume of a certain quantity of gas at absolute temperature 273 and under pressure 14 (pounds on the square inch) is 78 (cubic feet); find its volume at temperature 300 and under pressure 20.

Let v, t, p measure the volume of the gas, the absolute temperature and the pressure. Then

$$v \propto \frac{1}{p}, \text{ when } t \text{ is constant;}$$

also

$$v \propto t, \text{ when } \frac{1}{p} \text{ (or } p\text{) is constant.}$$

$$\therefore\ v \propto \frac{1}{p}.\,t$$

$$\therefore\ v = k.\frac{t}{p}$$

where k is constant for the quantity of gas in question.

If $t=273$ and $p=14$ it is given that $v=78$.

$$\therefore \quad 78=k\,.\,\frac{273}{14}.$$

$$\therefore \quad k=4\,.$$

$$\therefore \quad v=4\,.\,\frac{t}{p}.$$

Therefore if $t=300$ and $p=20$,

$$v=4\,.\,\frac{300}{20}=60.$$

The volume required is then 60 cubic feet.

In all problems like the preceding, in which the value of the constant is found, it is to be noted that, when magnitudes of different kinds appear, the value found depends upon the choice of the units of measurement.

EXERCISES

1. The volume of a pyramid is shewn in works on geometry to vary as (the area of) the base when the altitude is constant, and as the altitude when the base is constant. A certain pyramid whose base and altitude measure 18 and 5 is found to have 30 as the measure of its volume. Find the formula for the volume of a pyramid in terms of its base and altitude.

2. If v, h, r measure the volume, the altitude and the radius of the base of a cylinder, it is known that $v \propto h$ if r is constant, and $v \propto r^2$ if h is constant. For a cylinder in which $r=2$ and $h=5$ it is found that $v=62{\cdot}8320$. Find the formula for the volume of a cylinder, and also the volume of a cylinder for which $r=3$, $h=7$.

3. The weight of a coin of gold alloy varies as the square of its diameter, and as its thickness. When the thickness is 0·1cm. and the diameter is 2cm., the weight is 5·91g; find the weight of a coin of the same alloy of thickness 0·2cm. and diameter 3cm.

4. Given that y is equal to the sum of two quantities of which one is constant and the other varies as x, and that $y=1$ when $x=1$, while $y=-1$ when $x=2$, find

(1) y when $x=0{\cdot}5$;
(2) x when $y=0$.

5. If $x \propto \frac{1}{y}$ when z is constant, and $x \propto \frac{1}{z}$ when y is constant, where y and z are independent, then when y and z both vary $x \propto \frac{1}{yz}$.

here

EXAMPLES

Try the

1. The velocity (v) of a heavy particle falling freely from rest varies as the square root of the distance (s) through which the particle has fallen. If when $s = 16{\cdot}1$ (ft.) it is found that $v = 32{\cdot}2$ (ft. a sec.) find the equation connecting v and s.

2. The electrical resistance of a wire varies directly as the length of the wire and inversely as the square of the radius of the cross section. If, when a copper wire is 1m. long and 1mm. in diameter, the resistance is 0·02 ohms, find the resistance of a copper wire 7m. long and 1·5mm. in diameter.

3. It is shewn in works on geometry that the volume of a pyramid varies as its base when its height is constant and as its height when its base is constant. A cube may be divided into six equal pyramids, each having a face of the cube as base, and the centre of the cube as vertex.

From these facts obtain the general formula for the volume of a pyramid.

4. The value of w depends only on the values of the three independent variables x, y, z. If it is known that $w \propto x$ when y and z are constant, that $w \propto y$ when z and x are constant, and that $w \propto z$ when x and y are constant, shew that $w \propto xyz$, when x, y and z vary.

Illustrate by reference to the volume of a rectangular parallelopiped.

5. If $y \propto x$ shew that $x^2 + y^2 \propto xy$ and construct an illustration.

6. If x varies as y and inversely as the square root of z, and if, when y is 5 and z is 4, the value of x is 15, find the value of x when $y = 7$ and $z = 9$.

7. If y varies as the sum of two quantities, of which one is constant and the other varies as the square of x, and if $y = 4$ when $x = 1$, and $y = 28$ when $x = 5$, find y when $x = 3$.

8. Given that y varies as the sum of two variables, one of which varies as x and the other inversely as x, and that $y = 19$ when $x = 3$, and that $y = 11$ when $x = 2$, find y when $x = 1$.

9. If y equals the sum of three quantities of which one is constant, one varies as x, and the other inversely as x, and if $(y=9,\ x=1)$, $(y=8,\ x=2)$, $(y=9,\ x=3)$ are three sets of corresponding values, find y when $x=4$.

10. When a weight is hung by an elastic string, the *extension* (*i.e.*, the amount the string is stretched) varies as the weight and as the unstretched length of the string. If a weight of 3·5kg. stretches a string 1m. long to a length 1·1m., to what length will a weight of 4·5kg. stretch a string of the same kind of length 1·5m.?

11. When a weight is hung by an elastic string of given material, the extension varies as the weight, as the unstretched length of the string, and inversely as the square of the diameter of the string. If a weight of 8 lb. stretches a string 2·5 ft. long and of $\frac{1}{4}$ in. diameter to a length 2·7 ft., to what length will a weight of 10 lb. stretch a string of the same material 3·25 ft. long and of $\frac{1}{6}$ in. diameter?

12. If $x^2+y^2 \propto xy$ shew that $x \propto y$, and if, when $x^2+y^2=25$ it is known that $xy=12$, find the relation between x and y.

13. If y varies inversely as x, and if when $y=2$ it is known that $x=1$, construct a graph to exhibit related values of x and y.

CHAPTER IV

SERIES

A sequence of numbers as

$$1,\ 2,\ 3,\ 4,\ 5, \ldots\ldots$$

or

$$3,\ 7,\ 11,\ 15,\ 19, \ldots\ldots$$

or

$$3,\ 6,\ 12,\ 24,\ 48, \ldots\ldots$$

or

$$\frac{1}{1^2},\ \frac{1}{2^2},\ \frac{1}{3^2},\ \frac{1}{4^2},\ \frac{1}{5^2}, \ldots\ldots$$

in which the successive numbers are formed according to some law, is called a **series.**

As a rule, the numbers of a series will be connected, each to the preceding, by the sign + or −; and the individual numbers with their signs will be called **terms** of the series.

I

ARITHMETICAL SERIES

1. **Definition.** It is readily seen that

$$2+5+8+11+\ldots\ldots$$

is a series, each term being formed from the preceding term by the addition of 3. Thus, consecutive terms differ by the same number, or, in other words, the difference between consecutive terms is constant. Such a series is called an **arithmetical series,** or an **arithmetical progression** which may, therefore, be defined as follows:

An arithmetical progression is a series in which each term is formed from the preceding by the addition of the same quantity.

All arithmetical series are seen to be included in the *general* progression

$$a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots$$

Here a is called the **first term** and d (the difference between consecutive terms) the **common difference.**

Any term of a given arithmetical progression may be found without constructing all preceding terms. Thus, in the series

$$2+5+8+11+\ldots\ldots$$

in which 2 is the first term and 3 the common difference, it is readily seen that the *fiftieth* term, being *forty-nine* terms in advance of the first, is $2+3\times 49$ or 149, since each advance implies the addition of 3. So, too, the nth term of this series (n, a positive integer), being $n-1$ terms in advance of the first term, is $2+(n-1)3$ or $3n-1$. The nth term is usually spoken of as the **general term,** since from it any term can be found by giving to n a suitable value. Thus, here, the nth term being $3n-1$, the 5th term, for example, is $3\times 5-1$ or 14. In like manner we have for the general series,

$$n\textit{th term}=a+(n-1)d \qquad (I)$$

a result which should be remembered.

Ex. 1. The 5th term of an arithmetical progression is 19 and the 13th term is 43; find the progression and the 30th term.

Let a, d denote the first term and the common difference

$$\therefore \text{ The 5th term}=a+4d$$

$$\therefore \qquad a+4d=19.$$

Similarly, by constructing the 13th term we have

$$a+12d=43.$$

$$\therefore \text{ solving, } a=7,\ d=3.$$

Thus the series is

$$7+10+13+16+19+\ldots\ldots$$

and the 30th term $=7+3\times 29$ or 94.

Ex. 2. The nth term of a series is $3n+5$; shew that the series is an arithmetical progression.

Here n is any number and we may say that the 1st term is $3\times 1+5$ or 8, the 2nd term is $3\times 2+5$ or 11, the 3rd term is $3\times 3+5$ or 14, the 4th term is $3\times 4+5$ or 17, etc. It *appears* then that the series is an arithmetical progression of which the first term is 11 and the common difference is 3. But however many terms we construct we cannot say absolutely, on the evidence afforded by these terms, that for terms not constructed the law continues.

We may, however, reason generally and more briefly thus :

$$\text{The } n\text{th term} = 3n+5, \text{ whatever value be assigned to } n.$$

$$\therefore \text{ the } (n-1)\text{th term} = 3(n-1)+5.$$

Then the difference between the nth and the $(n-1)$th term is equal to

$$(3n+5)-3(n-1)-5 \text{ or } 3.$$

Now n is *any* number ; therefore the difference between *any* and therefore *every* two consecutive terms of the series is 3, so that the series is an arithmetical progression.

EXERCISES

(NOTE : In the exercises, A.P. will be employed as an abbreviation for arithmetical progression.)

1. Find the 47th and the nth term of the following series :

(1) $1+3+5+7+\ldots\ldots$
(2) $7+13+19+25+\ldots\ldots$
(3) $29+25+21+17+\ldots\ldots$
(4) $94+81+68+55+\ldots\ldots$

In each case after finding the nth term test the result by finding from it the first four terms.

2. The 7th term of an A.P. is 23, and the 15th term -17 ; find the series, its 23rd term, and its first negative term.

3. Find three numbers in A.P. such that the third is 7 times the first while the product of the first and the third exceeds 5 times the second by a quantity equal to the first.

4. How many multiples of 13 are there between 300 and 700 ?

5. Shew that the series formed by taking every fifth term of the A.P.

$$1+3+5+7+\ldots\ldots$$

is also an A.P.

6. Shew that the series formed by taking every seventh term of the A.P.

$$a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots$$

is also an A.P.

7. If to each term of an A.P. there be added the same number, the resulting series is an A.P.

8. If each term of an A.P. be multiplied by the same number the resulting series is an A.P.

9. Write down any 10 numbers in A.P. and shew that the average of the 1st and the 10th is the same as that of the 2nd and the 9th, or that of the 3rd and the 8th, etc.

10. Write down any 11 numbers in A.P. and shew that the average of the 1st and the 11th is equal to that of the 2nd and the 10th, or to that of the 3rd and the 9th, etc., and that this average is equal to the middle term.

11. The first of 17 terms of an A.P. is 5 and the last is 133; find the middle term.

12. The nth term of a series is $an+b$ whatever be the value of n; shew that the series is an A.P.

13. The pth term of an A.P. is q, and the qth term is p, where p and q are given integers; find the $(p+q)$th term.

2. **Arithmetical Means.** When three numbers are in arithmetical progression, the second number is called the **arithmetical mean** of the other two numbers. We may then have the problem:

To find the arithmetical mean of two given numbers.

Let a and b be the given numbers, and let x be the mean sought. Then by the definition

$$a,\ x,\ b$$

are in arithmetical progression, and therefore

$$\begin{aligned} x-a&=b-x\\ \therefore\quad 2x&=a+b\\ \therefore\quad x&=\frac{a+b}{2} \end{aligned}$$

so that *the arithmetical mean of two given numbers is equal to one-half their sum.*

In like manner we may have the problem of inserting any number of arithmetical means between two given numbers.

Ex. Insert 7 arithmetical means between 5 and 29.

Here 5, the 7 means sought, and 29 make up 9 terms in arithmetical progression. Let x be the common difference in this series of 9 terms. Then, the first term being 5, the ninth term equals $5+8x$, which must equal 29, *i.e.*,

$$\begin{aligned} 5+8x&=29\\ \therefore\quad 8x&=24\\ \therefore\quad x&=3 \end{aligned}$$

Thus the means are

$$8,\ 11,\ 14,\ 17,\ 20,\ 23,\ 26.$$

The general problem is:

To insert n arithmetical means between a and b.

Here a, the n means sought, and b make up $n+2$ terms in arithmetical progression. Let x be the common difference in this series of $n+2$ terms. Then, the first term being a, the $(n+2)$nd term is $a+(n+1)x$ which must be equal to b, *i.e.*,

$$\begin{aligned} a+(n+1)x&=b\\ \therefore\quad (n+1)x&=b-a\\ \therefore\quad x&=\frac{b-a}{n+1}. \end{aligned}$$

Thus the means are

$$a+\frac{b-a}{n+1},\ a+2\frac{b-a}{n+1},\ \ldots\ldots,\ a+(n-1)\frac{b-a}{n+1},\ a+n\frac{b-a}{n+1};$$

or, in simpler form,

$$\frac{na+b}{n+1},\ \frac{(n-1)a+2b}{n+1},\ \ldots\ldots\ \frac{2a+(n-1)b}{n+1},\ \frac{a+nb}{n+1}.$$

Since n has not any assigned value, it is not possible to write down all the means, but, the law of formation of successive means being known, we can regard all the means as found.

EXERCISES

1. Insert 11 arithmetical means between 19 and 223.

2. Insert 5 arithmetical means between 17 and -7.

3. When n arithmetical means are inserted between a and b find the rth mean.

4. Insert 5 arithmetical means between 11 and 25 and shew that the middle mean is the arithmetical mean of the given numbers.

5. Whatever odd number of arithmetical means be inserted between a and b shew that the middle mean is the arithmetical mean of a and b.

3. **The Summation Formula.** The sum of any number of terms of an arithmetical series may be found without actual addition.

Ex. Find the sum of 9 terms of the arithmetical progression,

$$13+17+21+\ldots\ldots.$$

Let s denote the sum sought. Then

$$s=13+17+21+25+29+33+37+41+45$$
$$\therefore\ s=45+41+37+33+29+25+21+17+13,$$

the terms being written in reverse order, which does not affect the sum. Therefore by addition

$$2s=58+58+58+58+58+58+58+58+58$$
$$\therefore\ 2s=58\times 9$$
$$\therefore\ s=\frac{58\times 9}{2}=261.$$

It is to be noted that, in adding to find $2s$, when we find for the first addition $13+45$ the result 58, we know beforehand that the successive additions will yield the same sum, because in the upper line the terms *increase* 4 at each advance while in the lower line the terms *decrease* 4 at each advance. We know also beforehand that there will be 9 addends 58, *i.e.*, one for each term of the series. It follows then that it is not necessary to write all the terms of the proposed addition.

The general problem is the following:

To find the sum of n *terms of the arithmetical progression*

$$a+(a+d)+(a+2d)+\ldots\ldots$$

Let l denote the last of the terms considered, *i.e.*, the nth term

$$\therefore\ l=a+(n-1)d.$$

Let s denote the sum sought :

$$\therefore \quad s=a+(a+d)+(a+2d)+\ldots\ldots+(l-2d)+(l-d)+l.$$

$$\therefore \quad s=l+(l-d)+(l-2d)+\ldots\ldots+(a+2d)+(a+d)+a,$$

the terms being written in reverse order which does not affect the sum. Therefore, by addition,

$$2s=(a+l)+(a+l)+(a+l)+\ldots\ldots+(a+l)+(a+l)+(a+l),$$

$$\therefore \quad 2s=(a+l)\times n,$$

since there is one term $a+l$ for each term of the series.

$$\therefore \quad s=\frac{n}{2}(a+l) \qquad (II)$$

Further, putting for l its value in terms of a, d, n, we have

$$s=\frac{n}{2}\left\{2a+(n-1)d\right\} \qquad (III)$$

The formulæ (II) and (III) should be given verbal statement, and committed to memory.

The following examples may be examined :

Ex. 1. How many terms of the series

$$21+35+49+\ldots\ldots$$

must be taken to yield the sum 336 ?

Let n be the number required. Then, the first term being 21 and the common difference being 14, we have, quoting the formula for the sum of n terms of an arithmetical series,

$$\frac{n}{2}\cdot\left\{2\times21+(n-1)14\right\}=336$$

$$\therefore \quad n(7n+14)=336$$

$$\therefore \quad n^2+2n-48=0$$

whence $n=6$ or -8. Plainly the result must be a positive integer so that the number sought is 6.

It is to be remarked that if the 6 terms which yield the sum 336 be written down, and if, starting at the last term, we write down 8 terms in the inverse order of the series, the sum of these 8 terms is 336. This is not an *interpretation* of the negative root -8, but an indication of a closely related problem in which the number $+8$ is significant.

Ex. 2. Shew that the sum of $5n$ terms of an arithmetical progression is 5 times that of the n terms beginning with the $(2n+1)$th term.

Let a, d be the first term and the common difference of the series.

$$\therefore \text{ sum of } 5n \text{ terms} = \frac{5n}{2}\cdot\left\{2a+(5n-1)d\right\} \qquad (1)$$

Also the $(2n+1)$th term $= a+2nd$

$\therefore$ sum of n terms beginning with the $(2n+1)$th

$$= \frac{n}{2}\cdot\left\{2(a+2nd)+(n-1)d\right\}$$

$$= \frac{n}{2}\cdot\left\{2a+(5n-1)d\right\} \qquad (2)$$

Then, comparing (1) and (2), we see that the result follows immediately.

EXERCISES

1. Sum to 53 terms, after the manner of the general theorem (*i.e.*, not quoting the formula), each of the following series:

(1) $5 + 11 + 17 + 23 + \ldots\ldots$

(2) $9 + 9\tfrac{3}{4} + 10\tfrac{1}{2} + 11\tfrac{1}{4} + \ldots\ldots$

(3) $117 + 112 + 107 + 102 + \ldots\ldots$

(4) $7\tfrac{1}{2} + 7\tfrac{1}{4} + 7 + 6\tfrac{3}{4} + \ldots\ldots$

✓ (5) $\dfrac{a}{b} + \dfrac{a+b}{b} + \dfrac{a+2b}{b} + \ldots\ldots$

2. Sum to n terms, after the manner of the general theorem, each of the following series:

(1) $9 + 13 + 17 + 21 + \ldots\ldots$

(2) $23 + 18 + 13 + 8 + \ldots\ldots$

(3) $2\tfrac{1}{2} + 3\tfrac{3}{4} + 5 + 6\tfrac{1}{4} + \ldots\ldots$

In each case *test* the result by assigning to n the value 4.

✓ 3. Shew that

(1) $1+2+3+4+5 \ldots\ldots$ to n terms $= \dfrac{n(n+1)}{2}$;

(2) $1+3+5+7+ \ldots\ldots$ to n terms $= n^2$.

The first of these series is that of the first n natural numbers, the second that of the first n odd numbers. The results are important and should be carried in memory.

4. In the general A.P. of n terms,

$$a+(a+d)+(a+2d)+\ .\ .\ .\ .\ .\ +(a+\overline{n-1}d)$$

Shew that the sum of the rth term from the beginning and the rth term from the end is equal to the sum of the first and the last term.

Hence shew that one-half the sum of the first and the last term is the average of the terms and derive the formula for the sum of n terms of an A.P.

5. How many terms of the series

$$44+36+28+\ .\ .\ .\ .\ .$$

must be taken to yield the sum 128?

Comment on the two results.

6. How many terms of the series

$$21+17+13+9+\ .\ .\ .\ .\ .$$

must be taken to yield the sum 65?

Comment on the fractional result.

7. How many terms of the series

$$20+17+14+\ .\ .\ .\ .\ .$$

must be taken to yield the sum 76?

Comment on the fractional root.

8. How many terms of the series

$$18+22+26+\ .\ .\ .\ .\ .$$

must be taken to yield the sum 210?

Comment on the negative root.

9. How many terms of the series

$$8+25+42+\ .\ .\ .\ .\ .$$

must be taken to yield the sum 1430?

Comment on the negative fractional root.

10. The nth term of a series is $5n+7$; shew that the series is an A.P., and find the sum of r terms.

11. The sum of 12 terms of an A.P. is 138 and the sum of 19 terms is 988; find the series and the sum of 26 terms.

12. Shew that the sum of an odd number of terms in A.P. is equal to the product of the middle term by the number of terms.

13. Find the sum of all multiples of 23 between 100 and 700.

14. The sum of n terms of a certain series is $7n^2+11n$, whatever be the value of n; find the rth term and shew that the series is an A.P.

15. Three integers are in A.P.; their sum is 24 and the product of the first and the last is 9 less than the square of the middle number. Find the numbers.

16. In a certain A.P. the first term is 29 and the last term 107; the sum of the terms is 952. Find the series and the number of terms.

17. Construct an A.P. such that the sum of 7 terms is equal to the sum of 11 terms, the common difference being 2.

18. If the pth, qth, rth terms of an A.P. are a, b, c respectively, shew that

$$(q-r)a+(r-p)b+(p-q)c=0.$$

19. A heavy particle, allowed to fall freely from a height, falls through 16·1 feet during the first second and in successive seconds falls through 32·2 feet more than during the preceding second. How far will it fall in t seconds?

II

GEOMETRICAL SERIES

1. **Definition.** It is readily seen that

$$2+6+18+54+\ldots\ldots$$

is a series, each term being formed from the preceding by multiplying it by 3. Thus consecutive terms stand to each other in a *constant* ratio. Such a series is called a **geometrical series,** or a **geometrical progression,** which may therefore be defined as follows:

A geometrical progression is a series in which each term is made from the preceding by multiplication by the same number.

All geometrical series are seen to be included in the general progression

$$a+ar+ar^2+ar^3+\ldots\ldots$$

Here a is called the **first term** and r (the ratio of any term to the preceding term) the **common ratio.**

Any term of a geometrical progression may be found without forming all preceding terms. Thus in the series

$$2+6+18+54+\ldots\ldots$$

in which 2 is the first term and 3 the common ratio, it is plain that the *fiftieth* term being *forty-nine* terms in advance of the first will be the product of 2 and the factor 3 taken forty-nine times, *i.e.*, will be 2.3^{49}, since each advance implies the introduction of the factor 3. So, too, the nth term of this series, being $n-1$ terms in advance of the first, is 2.3^{n-1}. As in the case of the arithmetical progression, the nth term is called the general term; thus, giving to n the value 4, we find the fourth term to be 2.3^3 or 54. In like manner we have for the general series,

$$nth\ term = ar^{n-1} \qquad (I)$$

a result which should be remembered.

Ex. 1. The 5th term of a geometrical progression is 162 and the 8th term is 4374; find the progression and the 10th term.

Let a, r denote the first term and the common ratio.

$$\therefore \quad \text{The 5th term} = ar^4$$

$$\therefore \quad ar^4 = 162.$$

Similarly, constructing the 8th term, we have

$$ar^7 = 4374.$$

Therefore, by division

$$r^3 = \frac{4374}{162} = 27$$

$$\therefore \quad r = 3$$

if we regard only the arithmetical cube root, or, in other words, if we consider only real numbers.

$$\therefore \quad a = \frac{162}{3^4} = 2.$$

The progression is then

$$2+6+18+54+162+\ldots\ldots$$

and the 10th term $= 2.3^9$

$= 39366.$

Ex. 2. The nth term of a series is 5.3^n; shew that the series is a geometrical progression and indicate the series.

Whatever be the value of n,

$$\text{The } n\text{th term} = 5.3^n$$

$$\therefore \text{ The } (n-1)\text{th term} = 5.3^{n-1}.$$

Therefore the ratio of the nth to the $(n-1)$th term is

$$\frac{5.3^n}{5.3^{n-1}}, \text{ or } 3.$$

Now n is *any* integer, so that the ratio of *any* and therefore of *every* two consecutive terms is 3, and the series is a geometrical progression.

In 5.3^n put $n=1$ and we find that the *first* term is 15. The series is therefore

$$15+45+135+405+\ldots\ldots$$

EXERCISES

(NOTE: In the exercises, G.P. will be employed as an abbreviation for geometrical progression.)

1. Find the 9th and the nth term of each of the following series:

(1) $1+5+25+\ldots\ldots$

(2) $7+14+28+\ldots\ldots$

(3) $2+\frac{2}{3}+\frac{2}{9}+\ldots\ldots$

(4) $1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots$

(5) $1-\frac{1}{2}+\frac{1}{4}-\ldots\ldots$

In each case after finding the nth term test the result by finding from it the first four terms.

2. The 11th term of a G.P. is $\frac{15}{32}$, and the 5th term is 30; find the series, and write down its 17th term.

3. Find three numbers in G.P. such that the third exceeds the sum of the other two by the first while the third exceeds the first by 36.

4. Shew that the series formed by taking every third term of the G.P.

$$7+21+63+\ldots\ldots$$

is also a G.P.

5. Shew that the series formed by taking every 5th term of the G.P.

$$a+ar+ar^2+\ldots\ldots$$

is also a G.P.

6. If each term of a G.P. be multiplied by the same number, the resulting series is a G.P.

7. If corresponding terms of two G.P. be multiplied together, and the results be written in order as terms, shew that the series is a G.P.

Illustrate this theorem.

8. Write down any 7 numbers in G.P. and shew that the average of the 1st and 7th is *not* the same as the average of the 2nd and 6th, or of the 3rd and 5th; but shew that the product of the 1st and 7th is equal to that of the 2nd and 6th, or that of the 3rd and 5th, and is equal to the square of the 4th.

9. The first of 17 terms of a G.P. is $\frac{1}{192}$ and the last is $341\frac{1}{3}$; find the middle term.

10. The nth term of a series ab^{2n}, whatever be the value of n; shew that the series is a G.P.

2. **Geometrical Means.** When three numbers are in geometrical progression, the middle number is called the geometrical mean of the other two. We have then the problem:

To find the geometrical mean of two given numbers.

Let a and b be the given numbers and let x be the mean sought. Then a, x, b are in geometrical progression and by definition

$$\frac{x}{a}=\frac{b}{x}$$

$$\therefore\ x^2=ab$$

$$\therefore\ x=\sqrt{ab}.$$

Thus, the geometrical mean of two numbers is equal to the square root of their product.

As it is supposed that we are dealing with *real* numbers the two numbers whose mean is sought are supposed to be of the same sign, and in extracting the square root of their product we take the sign so as to place the mean between the given numbers.

It is to be noted that, if three or more numbers are in geometrical progression, the numbers are in continued proportion, and that the geometrical mean of two numbers is the mean proportional of those numbers.

In like manner we have the problem of inserting any number of geometrical means between two given numbers.

Ex. Insert 5 geometrical means between $\frac{81}{64}$ and $\frac{1}{9}$. Here $\frac{81}{64}$, the 5 means sought, and $\frac{1}{9}$ make up 7 numbers in geometrical progression. Let r be the common ratio.

$$\therefore \text{ The 7th term} = \frac{81}{64} \cdot r^6$$

$$\therefore \frac{81}{64} \cdot r^6 = \frac{1}{9}$$

$$\therefore r^6 = \frac{64}{81 \times 9}$$

$$\therefore r = \frac{2}{3}$$

taking the positive real sixth root for reasons given. Therefore the means are

$$\frac{27}{32}, \frac{9}{16}, \frac{3}{8}, \frac{1}{4}, \frac{1}{6}.$$

We proceed in the same way in the case of the general problem:

To insert n geometrical means between a and b.

Here a, the n means sought, and b make up $n+2$ terms in geometrical progression. Let r be the common ratio

$$\therefore \text{ The } (n+2)\text{nd term} = ar^{n+1}$$

$$\therefore ar^{n+1} = b$$

$$\therefore r^{n+1} = \frac{b}{a}$$

$$\therefore r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

and the n means may be constructed.

EXERCISES

1. Insert 3 geometrical means between 7 and 567.

2. Insert 5 geometrical means between 5 and 588245.

3. Between 4 and 186624 a certain number of geometrical means have been inserted; the 3rd is 864 and the 5th is 31104. Find the means.

4. Whatever odd number of geometrical means be inserted between a and b, shew that the middle mean is the geometrical mean of a and b.

3. The Summation Formula. The sum of any number of terms of a geometrical series may be found without actual addition.

Ex. Find the sum of 8 terms of the series

$$7+21+63+\ldots\ldots$$

Let s denote the sum sought; then, since the common ratio is seen to be 3,

$$\begin{aligned} s &= 7+21+63+189+567+1701+5103+15309. \\ \therefore \quad 3s &= \quad 21+63+189+567+1701+5103+15309+45927. \end{aligned}$$

The second equation is formed from the first by multiplying each term by 3 and setting the result one place to the right. Then, since each term of the given series is formed from the preceding term by multiplying it by 3, we see *in advance* that each number in the series for $2s$ will be below an equal number. Then subtracting the numbers in the first equation from those in the second we have

$$2s = 45927 - 7 = 45920$$

$$\therefore \quad s = 22960.$$

Consider now the general problem:

To find the sum of n terms of the geometrical progression

$$a + ar + ar^2 + \ldots\ldots$$

Here the common ratio is r, so that the nth term is ar^{n-1}. Let s denote the sum sought. Then, having in mind the terms not written, we write

$$\begin{aligned} s &= a + ar + ar^2 + \ldots\ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \\ \therefore \quad rs &= \quad ar + ar^2 + \ldots\ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n. \end{aligned}$$

Then by subtraction we have

$$s - rs = a - ar^n$$

$$\therefore \quad s(1-r) = a(1-r^n)$$

$$\therefore \qquad \left. \begin{aligned} s &= a\frac{1-r^n}{1-r}, \\ \text{or, which is the same thing,} \qquad & \\ s &= a\frac{r^n-1}{r-1}. \end{aligned} \right\} \qquad (II)$$

These results should be carried in memory. As stated, they are equivalent, but it is more natural to employ the former when $r<1$, the latter when $r>1$.

Cor. From the results in (II) we find at once

$$\frac{1-r^n}{1-r}=\frac{r^n-1}{r-1}=1+r+r^2+\ldots\ldots+r^{n-1}$$

which may be regarded as affording a proof of the theorem that $1-r^n$, for all integral values of n, is divisible by $1-r$, the quotient being

$$1+r+\ldots\ldots+r^{n-1}.$$

EXERCISES

1. Sum to 29 terms, after the manner of the general theorem (*i.e.*, not quoting the result), each of the following series:

(1) $2+10+50+\ldots\ldots$

(2) $1+\frac{1}{7}+\frac{1}{49}+\ldots\ldots$

(3) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\ldots$

(4) $\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\frac{16}{81}+\ldots\ldots$

(5) $a+a^3+a^5+a^7+\ldots\ldots$

2. Sum to n terms, after the manner of the general theorem, each of the following series:

(1) $3+12+48+\ldots\ldots$

(2) $6+3+1\frac{1}{2}+\ldots\ldots$

(3) $1-\frac{1}{3}+\frac{1}{9}-\ldots\ldots$

In each case *test* the result by assigning to n the value 4.

3. Sum to n terms,

$$x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots\ldots$$

and point out the significance of the result.

4. In the general G.P. of n terms

$$a+ar+ar^2+\ldots\ldots+ar^{n-1}$$

shew that the product of the rth term from the beginning and the rth term from the end is equal to the product of the first and the last term.

5. Sum to n terms :

(1) $\left(1+\frac{1}{x}\right)^2+\left(1+\frac{1}{x^2}\right)^2+\left(1+\frac{1}{x^3}\right)^2+\ldots\ldots$

(2) $(a+b)^2+(a^2+b^2)^2+(a^3+b^3)^2+\ldots\ldots$

6. The product of three numbers in G.P. is 1728 and the sum of the first and the last is 25 ; find the numbers.

7. Divide the number 221 into three parts in geometrical progression such that the third exceeds the first by 136.

8. The sum of the first three terms of a G.P. is 228 and the sum of the first six terms 997·5 ; find the series.

9. Shew that if, in a G.P., each term be subtracted from the term following it, the successive differences form a G.P.

10. The sum of n terms of a certain series is $h(r^n-1)$; shew that the series is a G.P.

11. If the pth, qth, rth terms of a G.P. are a, b, c respectively, shew that

$$a^{q-r}.\ b^{r-p}.\ c^{p-q}=1.$$

12. Sum to n terms

$$9+99+999+\ldots\ldots$$

13. Sum to n terms

$$(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)+\ldots\ldots$$

4. **The Infinite Geometrical Series.** In the case of the geometrical series

$$1+2+2^2+\ldots\ldots$$

the nth term is 2^{n-1} and the sum of n terms is given by the formula

$$s_n=\frac{2^n-1}{2-1}=2^n-1,$$

where the subscript n in s_n indicates that it is a question of the sum of n terms. If to n be given the value 25 we find that the 25th term is 16,777,216 and that the sum of 25 terms is 33,554,431 and the rapidity with which the terms and the sums of terms increase with increase of n is very striking. It is easily seen that n may be taken sufficiently large to make either the nth term or the sum of n terms greater than any assigned number however large.

Consider next the geometrical series

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\dots\dots$$

Here the nth term, denoted by u_n, is

$$u_n=\frac{1}{2^{n-1}}$$

and the sum of n terms is

$$s_n=\frac{1-\dfrac{1}{2^n}}{1-\dfrac{1}{2}}=2-\frac{1}{2^{n-1}}.$$

If to n be given the value 25, we find that

$$u_{25}=\frac{1}{16,777,216},\quad s_{25}=2-\frac{1}{16,777,216},$$

and are struck by the smallness of the 25th term and by the proximity to 2 of the sum of 25 terms. From the expressions for u_n and s_n it is plain that, with increase of n, the value of u_n becomes smaller and smaller, and that the sum becomes more and more nearly equal to 2. It is plain too that n may be taken sufficiently large to make u_n less than any assigned positive quantity however small, and to make the sum, which is less than 2, differ from 2 by less than any assigned positive number, however small. Limiting the attention to s_n, *i.e.*, to the sum of n terms, we see that

(1) s_n is *always* less than 2;

(2) s_n increases with n;

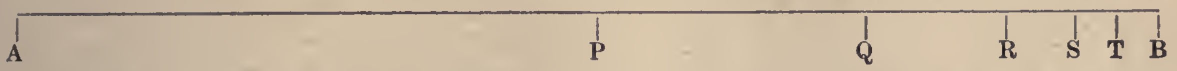
(3) n may be taken sufficiently large to make s_n approach 2 more nearly than by any assigned positive quantity however small, so that there is no number less than 2 that s_n cannot be made to exceed.

We say then that 2 is the **limit to the sum of n terms** as n is indefinitely increased, or, more briefly, that the **infinite series**

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\dots\dots$$

has 2 as its sum. The series may be looked upon as a perfectly definite though not the simplest way of giving the number 2.

That s_n may, by increasing n, be made to differ from 2 by as small a quantity as we please is rendered even more striking by representing the terms taken by lengths measured on a straight line.



AB measures 2; AP, 1; PQ, $\frac{1}{2}$; QR, $\frac{1}{4}$; RS, $\frac{1}{8}$; ST, $\frac{1}{16}$; etc. Then AQ measures the sum of 2 terms; AR of 3 terms; AS of 4 terms; AT of 5 terms; etc. Not many terms need be taken to make the sum practically 2, though the sum 2 is never reached.

The general series

$$a + ar + ar^2 + \ldots\ldots$$

may now be treated more concisely.

(I). Suppose r numerically greater than 1 or, in symbols, $|r| > 1$. Then denoted by s_n the sum of u terms we have

$$s_n = a\frac{r^n - 1}{r - 1} = a. \frac{r^n}{r - 1} - \frac{a}{1 - r}.$$

Now, r being numerically greater than 1, r^n increases in numerical value as n increases, and n may be taken sufficiently large to make r^n numerically greater than any assigned number however large. Therefore also, a and r being given values, n may be taken large enough to make

$$\frac{ar^n}{1 - r}, \text{ and consequently } \frac{ar^n}{1 - r} - \frac{a}{1 - r}$$

numerically greater than any assigned number. Thus, as n increases indefinitely, the numerical value of s_n tends beyond all limit and we may not in this case speak of the limit of the sum of n terms.

(II). Suppose $|r| < 1$.

Then,

$$s_n = a\frac{1 - r^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now, r being numerically less than 1, r^n decreases in numerical value as n increases, and n may be taken sufficiently large to make r^n numerically less than any assigned positive quantity, however small.

Therefore, also, a and r being given values, n may be taken large enough to make

$$\frac{ar^n}{1-r}$$

smaller than any assigned positive quantity, however small. Thus, as n increases indefinitely the numerical value of s_n differs less and less from that of $\frac{a}{1-r}$, and n may be taken large enough to make s_n differ from $\frac{a}{1-r}$ by less than any assigned positive number, however small.

We say then that:

The limit of the sum of n terms of the series

$$a+ar+ar^2+\ldots\ldots \qquad (|r|<1).$$

as n increases indefinitely is $\frac{a}{1-r}$,

or, in other words,

The sum of the infinite series

$$a+ar+ar^2+\ldots\ldots \qquad (|r|<1).$$

is $\frac{a}{1-r}$.

Ex. Find the value of $0{\cdot}\dot{5}$.

$$0{\cdot}\dot{5}=0{\cdot}55555\ldots\ldots \text{ in infinitum.}$$

$$=\frac{5}{10}+\frac{5}{100}+\frac{5}{1000}+\ldots\ldots \text{ in infinitum.}$$

This is an infinite G. P. whose common ratio is $\frac{1}{10}$, which is less than 1,

$$\therefore\quad 0{\cdot}\dot{5}=\frac{\frac{5}{10}}{1-\frac{1}{10}}=\frac{5}{9}.$$

EXERCISES

1. Write down the expression for the sum of n terms of each of the following series, and find the limit to the sum as n is indefinitely increased :

(1) $1+\frac{1}{3}+\frac{1}{9}+\ldots\ldots$

(2) $1+\frac{2}{3}+\frac{4}{9}+\ldots\ldots$

(3) $1-\frac{1}{2}+\frac{1}{4}-\ldots\ldots$

2. Write down the sums of the following infinite series :

(1) $5+\frac{10}{3}+\frac{20}{9}+\ldots\ldots$

(2) $2+\frac{6}{7}+\frac{18}{49}+\ldots\ldots$

(3) $1+\frac{1}{1\cdot05}+\frac{1}{(1\cdot05)^2}+\ldots\ldots$

3. Shew that any term of the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$$

is equal to the sum of the infinite series of succeeding terms.

4. Shew that any term of the infinite series

$$1+\frac{1}{3}+\frac{1}{9}+\ldots\ldots$$

is twice the sum of the infinite series of succeeding terms.

5. Construct the G.P. whose first term is 1 and in which any term is three-fourths the sum of the infinite series of succeeding terms.

6. Find the value of each of the following recurring decimals :

$$0\cdot\dot{7},\ 0\cdot\dot{0}1\dot{3},\ 0\cdot73\dot{2}1\dot{9}.$$

7. The middle points of the sides of a square are joined to form another square ; the middle points of this square are joined to form a new square ; and so on indefinitely. Find the sum of all the squares thus formed.

5. **Annuities.** An important application of the geometrical progression is found in the evaluation of **annuities.** An annuity is a payment recurring at the end of equal intervals for some stated time. Thus, if A is under obligation to pay B the sum of \$100 at the end of each year for the next 5 years, this recurring payment is called an annuity of \$100, **beginning** now and **running** for five years.

An annuity is said to be **deferred** for a number of years when it begins at the end of that number of years.

The interval between successive payments is generally one year, though it may be any period.

It is supposed known that the value of any stated sum of money depends upon where it is placed in time, *i.e.*, when it is to be paid or to become due, and upon the rate of interest. Thus, if \$400 is to be paid at the end of 3 years, and if the rate of interest is 5 *per cent.*, or 0·05 *on the unit* the present value of the payment is

$$\frac{\$400}{(1+0{\cdot}05)^3}, \text{ or } \frac{\$400}{(1{\cdot}05)^3},$$

and the value of the payment if not made until the end of 7 years, *i.e.*, 4 years after it is due, is

$$\$400 \times (1{\cdot}05)^4.$$

We can therefore say that the **present value** of the annuity cited in illustration, *i.e.*, of \$100 beginning now and running for 5 years, the rate of interest being given as 0·05 on the unit, is

$$\frac{\$100}{1{\cdot}05}+\frac{\$100}{(1{\cdot}05)^2}+\frac{\$100}{(1{\cdot}05)^3}+\frac{\$100}{(1{\cdot}05)^4}+\frac{\$100}{(1{\cdot}05)^5}.$$

This is seen to be a geometrical progression whose first term is $\frac{\$100}{1{\cdot}05}$ and common ratio is $\frac{1}{1{\cdot}05}$. The sum is therefore

$$\frac{\$100}{1{\cdot}05}\cdot\frac{1-\dfrac{1}{(1{\cdot}05)^5}}{1-\dfrac{1}{1{\cdot}05}}, \text{ or } \frac{\$100}{0{\cdot}05}\cdot\left\{1-\frac{1}{(1{\cdot}05)^5}\right\},$$

which is consequently the present value of the annuity in question.

The general problem is:

To find the present value of an annuity of $A beginning now and running for n years, the rate of interest being r on the unit.

Finding the present value of each payment, we have as the present value of all payments,

$$\frac{\$A}{1+r}+\frac{\$A}{(1+r)^2}+\cdots\cdots+\frac{\$A}{(1+r)^n},$$

a geometrical progression of n terms whose common ratio is $\frac{1}{1+r}$ and whose sum is therefore

$$\frac{\$A}{1+r}\cdot\frac{1-\dfrac{1}{(1+r)^n}}{1-\dfrac{1}{1+r}}$$

which reduces to

$$\frac{\$A}{r}\cdot\left\{1-\frac{1}{(1+r)^n}\right\}$$

which is consequently the present value sought.

If, in the preceding, we suppose n to increase indefinitely, which means that the payments are to continue for all time, we have then to do with an infinite geometrical series whose common ratio is $\frac{1}{1+r}$ which is less than 1, and the sum is

$$\frac{\dfrac{\$A}{1+r}}{1-\dfrac{1}{1+r}},$$

which reduces to

$$\frac{\$A}{r}.$$

That this is the present value is easily seen otherwise, for $\frac{\$A}{r}$ put out at interest for all time, the rate of interest being r on the unit would continue to bring in $\frac{\$A}{r}\times r$, or $A, each year for all time.

Such an annuity is called a **perpetuity.**

If an annuity of \$A, running for n years, is deferred for m years, the first payment is made at the end of $m+1$ years, so that its present value, r being the rate of interest on the unit, is seen to be

$$\frac{\$A}{(1+r)^{m+1}}+\frac{\$A}{(1+r)^{m+2}}+\cdots\cdots+\frac{\$A}{(1+r)^{m+n}}.$$

There are here n terms, and the series is a geometrical progression with common ratio $\frac{1}{1+r}$; the sum is therefore

$$\frac{\$A}{(1+r)^{m+1}}\cdot\frac{1-\dfrac{1}{(1+r)^n}}{1-\dfrac{1}{1+r}},$$

which reduces to

$$\frac{\$A}{r}\cdot\frac{1}{(1+r)^m}\cdot\left\{1-\frac{1}{(1+r)^n}\right\}.$$

EXERCISES

$50 . P_{7|}$
$\times 5.8927$

1. Find the present value of an annuity of \$250 to be paid at the end of each year for 7 years, money being worth $4\frac{1}{2}$ per cent. per annum.

$5 : P_{8|} -$
$5 : P_{3|}$

2. Find the present value of an annuity of \$375, deferred 3 years and running 5 years, the rate of interest being 4 per cent. per annum.

$375 (6.7327 - 2.7751) = 3.9576 \times 375.$

3. A man deposits \$500 at the beginning of each year for 7 years in a savings bank which allows 3 per cent. per annum compounded half-yearly on deposits. What sum will be standing to his credit at the end of the eighth year?

$P_{\infty|} = \frac{1}{i}$

4. If the rate of interest is 4 per cent., find the present value of a perpetuity of \$120.

5. Find the present value of a perpetuity of \$90, deferred 5 years, the rate of interest being 4 per cent.

6. A minor receives on account of his estate \$1200 on his sixteenth and each successive birthday. If interest is at $3\frac{1}{2}$ per cent., what sum should be charged against his estate just after he receives the payment on his twenty-first birthday?

III

HARMONICAL PROGRESSION

1. **Definition.** A series is said to be an **harmonical progression** when the series formed by the reciprocals of its terms is an arithmetical progression.

Let a, b, c be three numbers in harmonical progression. Then

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$$

are in arithmetical progression. Therefore

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$$

$$\therefore \quad \frac{a-b}{ab}=\frac{b-c}{bc}$$

$$\therefore \quad \frac{ab}{bc}=\frac{a-b}{b-c}$$

$$\therefore \quad \frac{a}{c}=\frac{a-b}{b-c}.$$

Thus, *if three numbers are in harmonical progression, the first is to the third as the difference between the first and the second is to the difference between the second and the third.*

This important property of three numbers in harmonical progression is sometimes taken as the definition of an harmonical progression of three terms.

Problems in harmonical progression may be solved by considering the analogous problem in arithmetical progression. This will be illustrated. It is to be remarked that there exists no formula for the sum of n terms of an harmonical progression.

Ex. Shew that 35, 45, 63 are in harmonical progression and find the nth term.

Consider $\frac{1}{35}, \frac{1}{45}, \frac{1}{63}$.

Then $\frac{1}{45}-\frac{1}{35}=-\frac{2}{315}$ and $\frac{1}{63}-\frac{1}{45}=-\frac{2}{315}$, so that $\frac{1}{35}, \frac{1}{45}, \frac{1}{63}$ are in arithmetical progression and therefore 35, 45, 63 in harmonical progression.

Next, the series

$$\frac{1}{35}+\frac{1}{45}+\frac{1}{63}+\ldots\ldots$$

being arithmetical with common difference $\frac{2}{315}$ will have for nth term

$$\frac{1}{35}+(n-1)\left(-\frac{2}{315}\right), \text{ or } \frac{11-2n}{315}.$$

Therefore the harmonical progression

$$35+45+63+\ldots\ldots$$

will have for nth term $\frac{315}{11-2n}$.

EXERCISES

(NOTE: In the exercises, H.P. will be employed as an abbreviation for harmonical progression.)

1. Shew that 15, 21, 35 are in H.P. and continue the series four terms.

2. The first two terms of an H.P. are 2 and 3; find the next two terms.

3. The 5th term of an H.P. is 168 and the 8th term is 108; find the first three terms.

4. The vertical angle C of a triangle ABC is bisected internally and externally by straight lines which meet the base in P and Q respectively; shew that AP, AB, AQ are in H.P.

5. If three numbers are defined to be in harmonical progression when the first is to the third as the difference between the first and the second is to the difference between the second and the third, and if a series is defined to be in harmonical progression when every consecutive three terms are in harmonical progression, shew that the reciprocals of the terms of an H.P. are in A.P.

2. **Harmonical Means.** If three numbers are in harmonical progression the middle number is called the **harmonical mean** of *the other two numbers;* we have then the problem:

To find the harmonical mean of a and b.

Let x be the mean sought. Then a, x, b are in harmonical progression and therefore $\frac{1}{a}$, $\frac{1}{x}$, $\frac{1}{b}$ are in arithmetical progression.

Therefore

$$\frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x}$$

$$\therefore \quad \frac{2}{x}=\frac{a+b}{ab}$$

$$\therefore \quad x=\frac{2ab}{a+b}.$$

Thus, *the harmonical mean of two numbers is the quotient of twice their product by their sum.*

If a and b are two positive numbers and if their arithmetical, geometrical and harmonical means be denoted by A, G, H, respectively, we may compare the values of A, G and H. For we have

$$A=\frac{a+b}{2};\ G=\sqrt{ab};\ H=\frac{2ab}{a+b}.$$

Then

$$AH=\frac{a+b}{2}\cdot\frac{2ab}{a+b}=ab=G^2$$

so that $G=\sqrt{AH}$ and G is not only the geometrical mean of a and b, but also the geometrical mean of A and H.

Again

$$\begin{aligned} A-G &= \frac{a+b}{2}-\sqrt{ab} \\ &= \frac{a-2\sqrt{ab}+b}{2} \\ &= \frac{(\sqrt{a}-\sqrt{b})^2}{2}. \end{aligned}$$

But, supposing a and b unequal, $(\sqrt{a}-\sqrt{b})^2$ being the square of a real number is positive, so that $A-G$ is positive or $A>G$. Further since $AH=G^2$ we have $\frac{A}{G}=\frac{G}{H}$, and A being greater than G we have $G>H$, so that $A>G>H$.

If $a=b$ it is readily seen that $A=G=H$.

EXERCISES

1. Insert two harmonical means between 10 and 20.

2. Find the arithmetical, the geometrical, and the harmonical mean of 3 and 12.

3. If the lines AB, BC measure x and y respectively, placing AB, BC in continuous straight line ABC, construct the arithmetical and geometrical means of x and y and shew that the former exceeds the latter unless x and y are equal.

4. If the harmonic mean of two numbers is to their geometric mean as 4 to 5, prove that the quantities are in the ratio of 1 to 4.

5. If the pth term of an H.P. is q and the qth term is p, where p and q are given integers, find the $(p+q)$th term.

6. The pth, qth, rth terms of an H.P. are a, b, c; prove that

$$(q-r)bc+(r-p)ca+(p-q)ab=0.$$

IV

ADDITIONAL IMPORTANT SERIES

In this section will be considered certain series whose sums can be obtained from the results found or by a modification of the methods employed in the preceding sections. We recall (Ex. 3, p. 60) the sum of the first n natural numbers:

$$1+2+3+\ldots\ldots+n=\frac{n(n+1)}{2}.$$

1. **The Squares of the Natural Numbers.** It is proposed tc find a formula for the sum of the squares of the first n natural numbers.

Denote the sum by S_n. Then

$$S_n=1^2+2^2+3^2+\ldots\ldots+n^2.$$

We have identically, *i.e.*, for all values of r,

$$r^3-(r-1)^3=3r^2-3r+1;$$

Hence, giving to r in succession the values $n, n-1, n-2, \ldots\ldots 2, 1,$ we have

$$\begin{array}{rcl} n^3 - (n-1)^3 &=& 3n^2 - 3n + 1, \\ (n-1)^3 - (n-2)^3 &=& 3(n-1)^2 - 3(n-1) + 1, \\ (n-2)^3 - (n-3)^3 &=& 3(n-2)^2 - 3(n-2) + 1, \\ \ldots\ldots\ldots\ldots &&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 3^3 - 2^3 &=& 3.3^2 - 3.3 + 1, \\ 2^3 - 1^3 &=& 3.2^2 - 3.2 + 1, \\ 1^3 - 0^3 &=& 3.1^2 - 3.1 + 1. \end{array}$$

Then, having regard to the lines not written but merely indicated by the dotted line, and noting that there are in all n lines, we have by addition

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \ldots\ldots + n^2) - 3(1 + 2 + 3 + \ldots\ldots + n) + n.$$

$$\therefore \quad n^3 = 3S_n - 3.\ \frac{n(n+1)}{2} + n$$

$$\therefore \quad 3S_n = n^3 + 3.\ \frac{n(n+1)}{2} - n$$

$$= \frac{n}{2}(2n^2 + 3n + 1)$$

$$= \frac{n(n+1)(2n+1)}{2}$$

$$\therefore \quad S_n = \frac{n(n+1)(2n+1)}{6}.$$

This result should be retained in memory.

Ex. Sum to n terms

$$1.3 + 2.5 + 3.7 + \ldots\ldots$$

Each term is the product of two factors. The *first* factors are $1, 2, 3, \ldots\ldots$ so that the nth *first* factor is n. The *second* factors are $3, 5, 7, \ldots\ldots$ so that the nth *second* factor is $3 + (n-1)2$ or $2n+1$. The nth term of the series is therefore $n(2n+1)$, which equals $2n^2 + n$. Now give to n the values $1, 2, 3, \ldots\ldots n$, and we have, denoting the sum sought by S_n,

$$
\begin{aligned}
S_n &= (2.1^2+1)+(2.2^2+2)+(2.3^2+3)+\dots\dots+(2.n^2+n) \\
&= \left\{ \begin{array}{l} 2(1^2+2^2+\dots\dots+n^2) \\ +\,(1+2+\dots\dots+n) \end{array} \right. \\
&= 2.\,\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1)}{3}+\frac{n(n+1)}{2},
\end{aligned}
$$

a formula which will give the sum of any number of terms The result may be brought to the simpler form

$$\frac{n(n+1)(4n+5)}{6}.$$

EXERCISES

Assuming the formulæ for the sum of the first n natural numbers and for the sum of their squares, find the sum to n terms of each of the following series, *testing* the result by assigning to n the values 1, 2, 3 :

1. $1^2+3^2+5^2+7^2+\dots\dots$
2. $1.2+2.3+3.4+4.5+\dots\dots$
3. $2^2+4^2+6^2+8^2+\dots\dots$
4. $3.7+5.10+7.13+9.16+\dots\dots$
5. $a^2+(a+b)^2+(a+2b)^2+(a+3b)^2+\dots\dots$

2. **The Cubes of the Natural Numbers.** It is proposed to find the sum of n terms of the series

$$1^3+2^3+3^3+\dots\dots$$

It is readily seen that the series is neither arithmetical nor geometrical, so that it calls for a special method of treatment.

Denote the sum sought by S_n so that

$$S_n=1^3+2^3+3^3+\dots\dots+n^3.$$

We have identically

$$r^4-(r-1)^4=4r^3-6r^2+4r-1.$$

Therefore, giving to r the values $n, n-1, n-2, \ldots\ldots 2, 1$, we have

$$\begin{aligned}
n^4 - (n-1)^4 &= 4n^3 - 6n^2 + 4n - 1, \\
(n-1)^4 - (n-2)^4 &= 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1, \\
(n-2)^4 - (n-3)^4 &= 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
2^4 - 1^4 &= 4.2^3 - 6.2^2 + 4.2 - 1, \\
1^4 - 0^4 &= 4.1^3 - 6.1^2 + 4.1 - 1.
\end{aligned}$$

Therefore, by addition, we have, since there are n lines,

$$\begin{aligned}
n^4 - 0^4 = 4(1^3 + 2^3 + \ldots\ldots + n^3) &- 6(1^2 + 2^2 + \ldots\ldots + n^2) \\
&+ 4(1 + 2 + \ldots\ldots + n) - n
\end{aligned}$$

$$\therefore \quad n^4 = 4\,S_n - 6.\frac{n(n+1)(2n+1)}{6} + 4.\frac{n(n+1)}{2} - n$$

$$\begin{aligned}
\therefore 4\,S_n &= n^4 + n - n(n+1)(2n+1) - 2n(n+1) \\
&= n(n+1)\left\{n^2 - n + 1 + 2n + 1 - 2\right\} \\
&= \left\{n(n+1)\right\}^2
\end{aligned}$$

$$\therefore \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Cor. $1^3 + 2^3 + \ldots\ldots + n^3 = (1 + 2 + \ldots\ldots + n)^2$.

Ex. Sum to n terms the series

$$1.3.5 + 3.5.7 + 5.7.9 + \ldots\ldots$$

As in the preceding article it is seen that the nth term is

$$(2n-1)(2n+1)(2n+3)$$

which is equal to $8n^3 + 12n^2 - 2n^2 - 3$.

Give to n the values $1, 2, 3, \ldots\ldots, n$, and, denoting the sum sought by S_n, we have

$$S_n=(8.1^3+12.1^2-2.1-3)+(8.2^3+12.2^2-2.2-3)+\ldots\ldots$$
$$+(8.n^3+12.n^2-2.n-3).$$

$$=\left\{\begin{array}{l} \quad 8(1^3+2^3+\ldots\ldots+r\ ) \\ +12(1^2+2^2+\ldots\ldots+n^2) \\ -\ \ 2(1\ +2\ +\ldots\ldots+n\ ) \\ -\quad (3\ +3\ +\ldots\ldots+3\ ) \end{array}\right\}$$

$$=8\ \left\{\frac{n(n+1)}{2}\right\}^2+12\ \frac{n(n+1)(2n+1)}{6}-2.\frac{n(n+1)}{2}-3n$$

$$=2n^2(n+1)^2+2n(n+1)(2n+1)-n(n+1)-3n$$

which may be reduced to the simpler form

$$n(2n^3+8n^2+7n-2).$$

EXERCISES

Sum to n terms each of the following series, testing the results by giving to n the values 1, 2, 3 :

(1) $1^3+3^3+5^3+7^3+\ldots\ldots$

(2) $1.2.3+2.3.4+3.4.5+\ldots\ldots$

(3) $2^3+4^3+6^3+8^3+\ldots\ldots$

(4) $1.3.5+2.5.8+3.7.10+\ldots\ldots$

(5) $(a+b)^3+(a+2b)^3+(a+3b)^3+\ldots\ldots$

3. **The Arithmetico-Geometric Series.** It is proposed to find the sum of n terms of the series

$$a+(a+b)r+(a+2b)r^2+\ \ldots\ldots$$

where each *term* is formed by multiplying corresponding terms of the arithmetical series

$$a+(a+b)+(a+2b)+\ \ldots\ldots$$

and the geometrical series

$$1+r+r^2+\ \ldots.$$

The nth term is seen to be $(a+\overline{n-1}b)r^{n-1}$. Denote the sum by S_n so that

$$S_n=a+(a+b)r+(a+2b)r^2+\ldots\ldots+(a+\overline{n-1}b)r^{n-1}$$

$$\therefore\ r.S_n=\qquad ar+\ (a+b)r^2+\ldots\ldots+(a+\overline{n-2}b)r^{n-1}+(a+\overline{n-1}b)r^n.$$

Then, by subtraction,

$$S_n(1-r)=a+(br+br^2+\ \ldots\ldots\ +br^{n-1})-(a+\overline{n-1}b)r^n$$

$$=a+br.\frac{1-r^{n-1}}{1-r}-(a+\overline{n-1}b)r^n,$$

since the series within the brackets is geometrical and consists of $n-1$ terms.

$$\therefore\quad S_n=\frac{a}{1-r}-\frac{(a+\overline{n-1}b)r^n}{1-r}+\frac{br(1-r^{n-1})}{(1-r)^2}.$$

Here the method is the important thing and the result need not be retained in memory.

EXERCISES

1. Sum to n terms

$$1+2x+3x^2+4x^3+\ldots\ldots$$

2. Sum to n terms

$$1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots\ldots$$

testing the result by giving to n the values 1, 2, 3.

3. Sum to n terms

$$7+12x+17x^2+22x^3+\ldots\ldots$$

4. Sum to n terms

$$1-2x+3x^2-4x^3+\ldots\ldots$$

EXAMPLES

1. Continue each of the following series three terms:

(1) 48, 60, 72;
(2) 48, 60, 75;
(3) 48, 60, 80.

2. The arithmetical mean of two numbers is 64 and their harmonical mean is 60; find the numbers.

3. The sum of four numbers in A.P. is 72 and the product of the extremes is to the product of the means as 27 to 35; find the numbers.

4. If the arithmetical mean between a and b is twice as great as the geometrical mean, shew that $a : b :: 2 + \sqrt{3} : 2 - \sqrt{3}$.

Obtain this result also geometrically.

5. If a, b, c are three given numbers, find the numbers which if added to each of them will give sums (1) in A.P., (2) in G.P., (3) in H.P.

6. If $2n+1$ terms of the series 1, 3, 5, 7, 9, ... be taken, shew that the sum of the alternate terms 1, 5, 9, ... will be to the sum of the remaining terms as $n+1$ to n.

7. On the ground lie n stones at intervals of 5 yards; how far will a person at the first stone have to travel to go and bring them one by one to the first stone?

8. On the ground lie n stones at intervals of 1 yard, 3 yards, 5 yards, 7 yards, etc.; how far will a person at the first stone have to travel to go and bring them one by one to the first stone?

9. Sum to n terms:

(1) $0.9 + 0.99 + 0.999 + \dots\dots$

(2) $0.7 + 0.77 + 0.777 + \dots\dots$

10. The series of natural numbers is divided into groups as follows:

$$1;\ 2,\ 3;\ 4,\ 5,\ 6;\ 7,\ 8,\ 9,\ 10;\ \text{etc.}$$

Find the sum of the numbers in the rth group.

11. Between a and b are inserted n geometrical means; find the sum of those means.

12. The sides of a right-angled triangle are in A.P.; shew that they are in the ratio of 3 : 4 : 5.

13. Sum to n terms:

(1) $1 + (1+b)r + (1+b+b^2)r^2 + \dots\dots$

(2) $(x+a) + (x^2+2a) + (x^3+3a) + \dots\dots$

(3) $1.1^2 + 2.3^2 + 3.5^2 + \dots\dots$

14. Sum to n terms:

$$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots$$

15. If a, b, c, d are in G.P., shew that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2.$$

Prove also the converse proposition.

16. If a^2, b^2, c^2 are in A.P., shew that $b+c$, $c+a$, $a+b$ are in H.P.

17. Shew that a, b, c are in A.P., G.P., or H.P., according as

$$\frac{a-b}{b-c} = \frac{a}{a}, \frac{a}{b}, \text{ or } \frac{a}{c}.$$

18. If the sum of n terms of a series is $a + bn + cn^2$ find the rth term and the nature of the series.

19. The sum of n terms of a certain A.P. is $(2n)^2$ for all values of n; find the series.

20. Sum to $2n$ terms

$$\frac{a}{r} - \frac{b}{r^2} + \frac{a}{r^3} - \frac{b}{r^4} + \ldots\ldots$$

21. If a, b, c are in H.P., shew that

(1) $a : a-b :: a+c : a-c$;

(2) $\dfrac{b+a}{b-a} + \dfrac{b+c}{b-c} = 2$;

(3) $\dfrac{1}{c-a} + \dfrac{1}{c-b} = \dfrac{1}{a} + \dfrac{1}{b}$.

22. If

$$x = a + \frac{a}{r} + \frac{a}{r^2} + \ldots\ldots \textit{ in inf.};$$

$$y = b - \frac{b}{r} + \frac{b}{r^2} - \ldots\ldots \textit{ in inf.};$$

$$z = c - \frac{c}{r^2} + \frac{c}{r^4} - \ldots\ldots \textit{ in inf.};$$

shew that $xy : z :: ab : c$.

23. If a, b, c are the pth, qth, rth terms of both an A.P. and a G.P., shew that

$$a^{b-c}b^{c-a}c^{a-b}=1.$$

24. If x, a_1, a_2, y are in A.P., x, g_1, g_2, y in G.P., and x, h_1, h_2, y in H.P. then

$$\frac{a_1+a_2}{h_1+h_2}=\frac{g_1g_2}{h_1h_2}\,;\quad xy=a_1h_2=a_2h_1\,;\quad a_1h_2+a_2h_1=2g_1g_2.$$

25. If x, a_1, a_2, a_3, y are in A.P., and x, h_1, h_2, h_3, y in H.P. shew that

$$xy=a_1h_3=a_2h_2=a_3h_1.$$

26. If S_n, S_{2n}, S_{3n} are the sums of n terms, $2n$ terms, $3n$ terms of a G.P., shew that

$$S_n(S_{3n}-S_{2n})=(S_{2n}-S_n)^2.$$

CHAPTER V

PERMUTATIONS AND COMBINATIONS

1. **Explanatory.** From the four letters a, b, c, d, it is plain that all possible *selections* of three are the following:

$$bcd,\ cda,\ dab,\ abc.$$

These are the **combinations** of four letters three at a time, and they are four in number.

Take any one of these combinations, say bcd; then by interchange of the letters b, c, d, the only *arrangements* of three that can be formed of those letters are the following:

$$bcd,\ bdc,\ cdb,\ cbd,\ dbc,\ dcb.$$

Each of the remaining three combinations will give rise to six such arrangements, so that in all there can be formed twenty-four arrangements of three letters. These are the **permutations** of four letters three at a time.

So, too, we speak generally of the combinations and permutations of *n things r at a time.* The things are ordinarily denoted by letters, different letters denoting dissimilar things and like letters like things. When nothing to the contrary is stated, the things will be supposed to be dissimilar. In the theorems and problems to be treated it will always be a question of the *number* of possible permutations or combinations in question.

EXERCISES

1. Determine the number of combinations and permutations of three letters, (1) one at a time, (2) two at a time, (3) three at a time, by actually forming the combinations and permutations.

2. Determine the number of combinations and permutations of four letters, (1) one at a time, (2) two at a time, (3) three at a time, (4) four at a time, by actually forming the combinations and permutations.

3. Given that the number of permutations of 5 things 3 at a time is 60, shew that the number of combinations of 5 things 3 at a time is 10 and that the number of permutations of 5 things 4 at a time is 60 × 2 or 120.

2. **The Fundamental Theorem of Permutations.** The theorem will first be illustrated by finding the number of permutations of 5 letters a, b, c, d, e, taken 3 at a time.

The number of ways in which three places, in order as shewn in the diagram,

b	d	e

may be filled by 3 of the letters is plainly the number of permutations sought. The first place may be filled by any one of the 5 letters and therefore in 5 ways. Suppose the first place filled in any of the 5 ways; then the second place may be filled by any one of the 4 remaining letters and therefore in 4 ways. Thus each of the 5 possible ways of filling the first place may be associated with each of the 4 possible ways of filling the second, which makes in all 5.4 ways of filling the first two places. Suppose the first two places filled in any way; the third place may then be filled by any one of the 3 remaining letters. As before, each of the 5.4 ways of filling the first two places may be associated with 3 ways of filling the third, which makes in all 5.4.3 ways of filling the three places. Hence, 5.4.3 or 60 is the number of permutations sought.

One way of filling the places is indicated. The student is recommended to write out all the ways of filling the second after b has been put in the first, and then to write out all the ways in each case for filling the third place.

The general proposition is: *To find the number of permutations of n different things r at a time.*

The number of ways of filling r places in order, each place by one thing, is the number of permutations sought.

The first place may be filled by any one of the n things and therefore in n ways. Suppose it filled in *any* one way; the second place may then be filled by any one of the $n-1$ remaining things and therefore in $n-1$ ways. Each of the n ways of filling the first may thus be associated with $n-1$ ways of filling the second which makes

$n(n-1)$ ways of filling the first two places. In like manner each of the $n(n-1)$ ways of filling the first two places may be associated with $n-2$ ways of filling the third which makes $n(n-1)(n-2)$ ways of filling the first three places. The reasoning may evidently be continued. When $r-1$ places have been filled, there remain $n-\overline{r-1}$ or $n-r+1$ things so that the rth place may be filled in $n-r+1$ ways. Thus the r places may be filled in

$$n(n-1)(n-2)\ldots\ldots(n-r+1)$$

ways, and this then is the number of permutations of n things r at a time. The number is frequently denoted by the significant symbol ${}_nP_r$.

Cor. The number of permutations of n *things* n *at a time, i.e., all together is*

$$n(n-1)\ldots\ldots 3.2.1.$$

This number is generally denoted by the symbol $\underline{|n}$ or $n\,!$ which is read n **factorial.**

Since

$$n(n-1)\ldots\ldots(n-r+1)=\frac{n(n-1)\ldots\ldots(n-r+1)(n-r)(n-r-1)\ldots\ldots 2.1}{(n-r)(n-r-1)\ldots\ldots 2.1}$$

we have

$${}_nP_r=n(n-1)\ldots\ldots(n-r+1)=\frac{n\,!}{(n-r)\,!}.$$

Ex. How many numbers of three digits, all different, may be formed from the digits 1, 2, 3, 4, 5?

This is readily seen to be the same as the number of permutations of 5 things 3 at a time and is therefore

$$5.4.3 \text{ or } 60.$$

EXERCISES

1. In how many ways may 5 books be arranged on a shelf?

2. Write down the numbers represented by ${}_{17}P_{13}$, ${}_{12}P_{7}$, ${}_{8}P_{8}$ and compute the value in each case.

3. How many words, each of four letters, may be formed from the letters of the word *comrade?*

4. How many numbers, each of 4 digits, may be made from the nine digits, no digit being employed more than once?

In how many of these will the first digit be an odd digit, and in how many of them will the first digit be 6?

5. How many numbers, each of 4 digits, may be made from the nine digits and the figure 0, no digit being employed more than once?

How many of these will end in 0?

6. How many numbers of 4 figures each may be made from the nine digits, if digits may be repeated?

7. A signal is made by running up on a vertical rope one, two, or three flags. How many signals could be made with 7 flags of different colours?

8. How many words each of four letters, beginning and ending with a consonant, may be made from the letters of the word *tambour?*

9. In how many ways may 5 ladies and 5 gentlemen be assigned to 10 seats in a row, no two ladies to be seated together?

10. Find in how many ways 9 persons may be seated at a round table

(1) supposing the seats distinguished;
(2) considering relative position only;
(3) considering relative position only except that two orders differing only in *direction* (or *sense*) are counted as one.

11. In how many ways may 11 persons be seated relatively at a round table if a certain two persons are not to be placed together?

12. In how many ways may 5 ladies and 5 gentlemen be seated relatively at a round table if no two ladies are to be seated together?

3. **The Fundamental Theorem of Combinations.** The theorem will be illustrated by finding the number of combinations of 5 letters a, b, c, d, e, taken 3 at a time. Denote the number sought by N. Take any combination abc; then by interchange of these letters we obtain in all 3.2.1 distinct permutations, each of three letters. The same is true of each of the possible combinations. Now no two distinct combinations can give rise to the same permutation, while from all possible combinations will be formed all possible permutations. Hence the total number of permutations of 5 letters 3 at a time is

$$N \times 3.2.1.$$

But this number is already known to be 5.4.3.

$$\therefore \ N \times 3.2.1 = 5.4.3$$

$$\therefore \ N = \frac{5.4.3}{3.2.1} = 10.$$

The student is recommended to write down these 10 combinations and *see* their relation to the 60 permutations of 5 letters 3 at a time.

The general position is: *To find the number of combinations of n things r at a time.*

Denote this number by ${}_nC_r$. Take any combination of the r things; then, by interchange of the r things of which it consists, it will give rise to $r\,!$ distinct permutations. The same is true of each combination. Further, no two combinations can give rise to the same permutation, while from all possible combinations will be formed all possible permutations. Therefore, the total number of permutations of n things r at a time is

$$ {}_nC_r \times r\,! $$

But this number is already known to be

$$ n(n-1)\,.\,.\,.\,.\,.\,(n-r+1), \text{ or } \frac{n\,!}{(n-r)\,!}. $$

$$ \therefore\ {}_nC_r \times r\,! = n(n-1)\,.\,.\,.\,.\,.\,(n-r+1) = \frac{n\,!}{(n-r)\,!}. $$

$$ \therefore\ {}_nC_r = \frac{n(n-1)\,.\,.\,.\,.\,.\,(n-r+1)}{r\,!} = \frac{n\,!}{r\,!\,(n-r)\,!}. $$

Cor. The number of combinations of n things r at a time is equal to the number of combinations of n things n − r at a time.

EXERCISES

1. From a company of 35 soldiers a picket of 5 soldiers has to be chosen; in how many ways is this possible?

2. Write down in full, *i.e.*, not employing the factorial symbol, the numbers denoted by ${}_7C_3$, ${}_9C_5$, ${}_5C_5$, and compute the value in each case.

3. From a case containing 15 books a person is to select 3 books; in how many ways may the selection be made?

4. If ${}_nC_{11} = {}_nC_7$ find n.

5. In a plane are 7 points; how many triangles may be formed with 3 of these points as angular points?

6. How many triangles can be formed having 3 of n given points in a plane as angular points?

7. From a committee of 15 ladies and 20 gentlemen, a sub-committee of 3 ladies and 4 gentlemen is to be chosen; in how many different ways may this be done?

8. How many diagonals has a heptagon? a quindecagon?

9. From 8 teachers and 60 pupils a committee of 3 teachers and 7 pupils is to be chosen; in how many ways may this be done?

In how many ways may the committee be chosen if a certain teacher and a certain 2 pupils are to serve on it?

10. Show that

$$ {}_nC_r = \frac{n-r+1}{r} \cdot {}_nC_{r-1} $$

11. In how many ways may 15 different things be divided among 3 persons, each getting 5 things?

12. In how many ways may 15 different things be divided into 3 parcels of 5 each?

13. In how many ways may 10 different things be divided among A, B, C, D, if A and B are each to receive 2 things and C and D each 3 things?

14. In how many ways may 10 different things be divided into two parcels of 2 each and two parcels of 3 each?

15. Shew that

$$ {}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r. $$

16. In a plane are n points, p of which are in a straight line; how many triangles can be formed with 3 of the points as angular points?

17. Shew that

$$ \frac{(2n)!}{n!} = 2^n(1.3.5.\dots.\overline{2n-1}). $$

18. In a plane are n points and through every two of them a straight line (produced indefinitely) is drawn. Find the number of points of intersection of these lines, exclusive of the n points.

19. How many combinations of 3 letters may be made from the letters of the word *Canada?*

20. How many combinations of 4 letters may be made from the letters of the word *Manhattan?*

21. Shew that mn different things may be divided among m persons, each receiving n things, in

$$\frac{(mn)!}{(n!)^m}$$

ways and that the number of ways in which the mn things may be divided into m parcels of n each is

$$\frac{(mn)!}{(n!)^n m!}$$

22. How many words of 3 consonants and 2 vowels may be formed from the letters of the word *amplitude?*

In how many of these will the vowels be separated?

23. Shew that the product of any r consecutive integers is divisible by r.

4. **Additional Theorems.** The theorems of the two preceding articles have been spoken of as fundamental. As the theorem on combinations has been derived from that on permutations, we might say that there is one essential proposition. We shall now derive certain further theorems either from those already given or by the methods already employed.

Theorem I. *The number of permutations of* n *things taken all at a time,* p *being alike and the rest unlike,* is

$$\frac{n!}{p!}.$$

Let N denote the number sought. Take any one of the permutations and in it suppose the p like things replaced by p things different from one another and from the rest. Then by permitting these p things without disturbing the rest we should form $p!$ distinct permutations. The same is true of each of the N permutations in question so that these would give rise to $\mathrm{N} \times p!$ permutations. But these would be the $n!$ permutations among themselves of n unlike things.

$$\therefore \quad \mathrm{N} \times p! = n!$$

$$\therefore \quad \mathrm{N} = \frac{n!}{p!}$$

Cor. The number of permutations of n things taken all at a time, p being of one kind, q of another kind, r of another kind, is

$$\frac{n!}{p!\,q!\,r!\ldots\ldots}$$

Theorem II. *The number of permutations of n things r at a time, when repetitions of each thing are allowed up to r times,* is n^r.

As in the earlier theorem suppose that there are r places to be filled each by one thing. The first may be filled in n ways; after it has been filled in any way the second may be filled in n ways, since the thing already placed may be repeated. Thus the number of ways of filling 2 places is n^2 and by continuing the reasoning we find that the number of ways of filling r places is n^r.

Theorem III. *The total number of ways in which a selection of one or more things from n things may be made is* $2^n - 1$.

The first thing may be taken or left, *i.e.*, it may be treated in 2 ways. In either case the second thing may be treated in 2 ways, so that the first two things may be treated in 2^2 ways. Similarly for the successive things and the number of ways of treating all the things is 2^n. As this includes the case in which all the things are left, and therefore no selection made, the result sought is $2^n - 1$.

Theorem IV. *The value of r for which the number of combinations of n things r at a time is greatest is* $\frac{n}{2}$ *if n is even, and* $\frac{n-1}{2}$ *or* $\frac{n+1}{2}$ *when n is odd.*

It is readily seen that

$$_n\mathrm{C}_r = {_n\mathrm{C}_{r-1}} \times \frac{n-r+1}{r}.$$

Therefore,

$$\begin{aligned} &{_n\mathrm{C}_r} >=< {_n\mathrm{C}_{r-1}} \\ \text{as}\quad &\frac{n-r+1}{r} >=< 1\ ; \\ \therefore\ \text{as}\quad &n-r+1 >=< r\ ; \\ \therefore\ \text{as}\quad &n+1 >=< 2r\ ; \\ \therefore\ \text{as}\quad &2r <=> n+1\ ; \\ \therefore\ \text{as}\quad &r <=> \frac{n+1}{2}. \end{aligned}$$

(*a*) Let n be even. Then the greatest value the integer r can have to be less than $\frac{n+1}{2}$ is $\frac{n}{2}$; this, therefore, is the greatest value r can have if ${}_nC_r > {}_nC_{r-1}$. Thus, ${}_nC_r$ is greatest when $r = \frac{n}{2}$.

(*b*) Let n be odd. Then the greatest value r can have to be less than $\frac{n+1}{2}$ is $\frac{n-1}{2}$; this, therefore, is the greatest value r can have if ${}_nC_r > {}_nC_{r-1}$. Thus ${}_nC_r$ is greatest when $r = \frac{n-1}{2}$. But we must note, also, that if $r = \frac{n+1}{2}$, *i.e.*, greater by 1 than $\frac{n-1}{2}$, then will ${}_nC_r = {}_nC_{r-1}$ so that the number of combinations $\frac{n-1}{2}$ at a time is the same as the number $\frac{n+1}{2}$ at a time and each is greater than the number given by any other value of r.

EXAMPLES

1. How many numbers of 5 figures each may be made from the nine digits

(1) if digits may not be repeated;

(2) if digits may be repeated?

In (1) how many numbers begin with 5 and end with 7, and how many have 5 as their middle digit?

In (2) how many numbers begin and end with 5, and how many have 5 as their middle digit?

2. Find the total number of possible combinations of $p+q$ things of which p are of one kind and q of another.

3. Find the number of combinations of 4 letters and the number of permutations of 4 letters that can be formed from the letters of the word *terrestrial*.

4. Shew that the number of combinations, n at a time, of $2n$ things of which n are alike exceeds by 1 the total number of combinations of n things.

5. In a city are m streets running north and south and n streets running east and west. Find in how many different ways the direct journey from the north-east corner of the city to the south-west corner may be made.

6. If twenty persons are seated at a round table, in how many ways may three of them be selected if no two of these three are to be seated together?

7. In how many ways can p positive signs and n negative signs be arranged in a straight line if no two negative signs are to be together?

8. Find in how many ways 8 like things may be given to 5 persons

(1) if there is made no restriction as to the mode of distribution;

(2) if each person is to receive at least one thing;

(3) if each person is to receive one thing and no person more than two.

CHAPTER VI

THE BINOMIAL THEOREM

1. **Preliminary.** By actual multiplication we know the expansion of $(a+x)^2$, $(a+x)^3$, $(a+x)^4$, and it may be a few higher powers of the *binomial* $a+x$. Thus, for example,

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

The question arises whether it is possible to obtain a rule for writing down the expansion of a binomial to *any positive integral* power and whether we can speak of an expansion and a rule for an expansion if the exponent is a *positive fraction* or a *negative integer or fraction.*

It will be seen that a rule is furnished by the **Binomial Theorem.**

The case first to be examined is that in which the exponent is a positive integer. It will be well, before treating the general problem, to obtain the expansion of say $(a+x)^3$ by a method other than *formal* multiplication. Plainly $(a+x)^3$ is the product of the three factors,

$$a+x,\ a+x,\ a+x.$$

Each term of the product will be of three dimensions, one dimension or letter being furnished by each factor, so that all possible terms are

$$a^3,\ a^2x, ax^2,\ x^3$$

except that coefficients are wanting. The term a^3 can and will be formed in only one way, namely by taking a from each factor; its coefficient is therefore 1. The term a^2x will be formed by taking x from one factor and with it the a from each of the other two factors; x can be chosen from one of three factors in 3 ways so that the term a^2x can and will be formed in 3 ways and its coefficient is therefore 3. The term ax^2 will be formed by taking x from two factors, and with these two x's the a from the remaining factor; x can be chosen from two of three factors in $\dfrac{3.2}{1.2}$ or 3 ways and 3 is the coefficient of ax^2. The coefficient of x^3 is seen to be 1. Thus we have

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

EXERCISES

1. Find in the manner just given the expansions of

$$(a+x)^4,\ (a+x)^5,\ (a+x)^6.$$

2. Find in like manner the expansions of

$$(a-x)^3,\ (a+2b)^5,\ (2a-3b)^7.$$

3. How many terms in the expansions of

$$(a+x)^9,\ (a-x)^{15},\ (2a+3b)^{17}?$$

2. **The Binomial Theorem for a Positive Integral Exponent.** It is proposed to find the expansion of $(a+x)^n$ where n is a positive integer.

By $(a+x)^n$ is meant the product of n factors

$$a+x,\ a+x,\ \ldots\ldots\ a+x.$$

Each term will be of n dimensions, one dimension or letter coming from each factor, so that all possible terms are

$$a^n,\ a^{n-1}x^1,\ \ldots\ldots\ a^{n-r}x^r,\ \ldots\ldots\ x^n,$$

except that the coefficients remain to be found.

The term a^n can and will be formed in only one way, namely, by taking a from each factor; hence its coefficient is 1.

The term $a^{n-1}x$ will be formed by taking x from any one factor with the a from each of the remaining factors; as x can be chosen in n ways, the coefficient of $a^{n-1}x$ is n.

The *general* term $a^{n-r}x^r$, the $(r+1)$th in order, will be formed by taking x from any r of the factors with the a from each of the remaining factors; as the r factors which are to furnish x can be chosen in ${}_nC_r$ ways, the coefficient of $a^{n-r}x^r$ is

$${}_nC_r, \text{ or } \frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}, \text{ or } \frac{n!}{(n-r)!\,r!}.$$

By giving to r the values $1, 2, \ldots\ldots n$, we obtain the coefficients of all the terms after the first. Hence we have

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \ldots\ldots$$
$$+ \frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}a^{n-r}x^r + \ldots\ldots + x^n,$$

and the required rule for the expansion has been found.

Cor. 1. The sum of the coefficients in the expansion of $(a+x)^n$ *is* 2^n. (Found by putting $a=1$, $x=1$.)

Cor. 2. The sum of the odd coefficients is equal to the sum of the even coefficients. (Found by putting $a=1$, $x=-1$.)

Cor. 3. The coefficients of terms equidistant from the beginning and the end are the same.

NOTE:—

(1) The number of terms is $n+1$, so that if n is even there is a middle term, and if n is odd there are two middle terms.

(2) For convenience the expansion is frequently written thus:

$$(a+n)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \ldots\ldots + \binom{n}{r}a^{n-r}x^r + \ldots\ldots + x^n,$$

where $\binom{n}{r}$ denotes $\dfrac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}$. We may agree to denote the first and the last coefficient, namely 1, by $\binom{n}{0}$.

(3) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots + \binom{n}{r}x^r + \ldots\ldots + x^n.$

(4) $(a-x)^n = a^n - \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 - \ldots\ldots + (-1)^r\binom{n}{r}a^{n-r}x^r + \ldots\ldots + (-1)^n x^n.$

Ex. 1. Find the middle term of $(2x-3y)^{10}$.

There are in all 11 terms so that the middle term is the 6th, *i.e.*, the term involving y^5. It is therefore

$$\frac{10.9.8.7.6}{1.2.3.4.5}(2x)^5(-3y)^5, \text{ or } -252.32.243x^5y^5.$$

Ex. 2. Find the sum of the squares of the coefficients in the expansion of $(1+x)^n$.

Denote the coefficient of x^r by c_r; then $c_0=1=c_n$ and $c_r=c_{n-r}$.

$$\therefore\ (1+x)^n = c_0 + c_1\ x + c_2\ x^2 + \ldots\ldots + c_{n-1}x^{n-1} + c_n x^n.$$

$$\therefore\ (1+x)^n = c_n + c_{n-1}x + c_{n-2}x^2 + \ldots\ldots + c_1\ x^{n-1} + c_0 x^n.$$

In the product of the series the coefficient of x^n is

$$c_0{}^2 + c_1{}^2 + c_2{}^2 + \ldots\ldots + c_n{}^2$$

and this then is equal to the coefficient of x^n in $(1+x)^{2n}$ and therefore to $\dfrac{(2n)!}{n!\ n!}$.

The student is recommended to work the example by finding the coefficient of x^0 in the product of the expansions of $(1+x)^n$ and $\left(1+\dfrac{1}{x}\right)^n$.

Ex. 3. If

$$(1+x)^m = a_0 + a_1 x + \ldots\ldots + a_r x^r + \ldots\ldots + a_m x^m,$$

$$(1+x)^n = b_0 + b_1 x + \ldots\ldots + b_r x^r + \ldots\ldots + b_n x^n,$$

find the value of

$$b_0 a_r + b_1 a_{r-1} + b_2 a_{r-2} + \ldots\ldots + b_r a_0.$$

This is evidently the coefficient of x^r in the product of the two series, and, therefore, to the coefficient of x^r in $(1+x)^{m+n}$ which is equal to

$$\frac{(m+n)(m+n-1)\ldots\ldots(m+n-r+1)}{1.2\ldots\ldots r} \text{ or } \binom{m+n}{r}.$$

$$\therefore \binom{m}{r}\binom{n}{0}+\binom{m}{r-1}\binom{n}{1}+\binom{m}{r-2}\binom{n}{2}+\ldots\ldots+\binom{m}{0}\binom{n}{r}=\binom{m+n}{r}.$$

EXERCISES

1. Write out the complete expansions of :

$$(a+b)^8,\ (a-b)^7,\ (a+2b)^5,\ (1-\tfrac{1}{2}x)^6,\ (1+\tfrac{2}{3}x)^4.$$

2. Find the middle term of the expansions of :

$$(x+y)^6,\ (x-y)^{20},\ (2a-3y)^{14},\ (\tfrac{1}{2}x-\tfrac{1}{3}y)^{12}.$$

3. Find the two middle terms of :

$$(2a-3b)^5,\ (x-y)^{23},\ (x^2-y^2)^{17}.$$

4. Write down the general term, *i.e*, the $(r+1)$th term, of :

$$(1-x)^n,\ (1+3x)^m,\ (a-2x)^n.$$

3. The Binomial Theorem for Fractional or Negative Exponents. The question now arises whether an expansion can be found for $(a+x)^n$ when n is fractional or negative, and, if so, whether the rule established for the case in which the exponent is a positive integer is applicable. We shall first examine certain examples.

Ex. 1. Examine whether an expansion in ascending powers of x can be found for $(1-x)^{-1}$.

We have $(1-x)^{-1}=\dfrac{1}{1-x}=1\div(1-x)$.

Let the division be performed :

$$\begin{array}{llll}
1-x)1 & & & (1+x+x^2+\ldots. \\
\quad\underline{1-x} & & & \\
\qquad +x & & & \\
\qquad \underline{+x-x^2} & & & \\
\qquad\qquad +x^2 & & & \\
\qquad\qquad \underline{+x^2-x^3} & & & \\
\qquad\qquad\qquad +x^3 & & &
\end{array}$$

Plainly the operation of division will not terminate. At any step we may write down a value of $(1+x)^{-1}$, *in part* a series. Thus we may say

$$(1-x)^{-1}=1+x+x^2+\frac{x^3}{1-x}.$$

We are led to say

$$(1-x)^{-1}=1+x+x^2+x^3+\ldots\ldots \textit{in inf.}$$

And we have indeed seen that the limit of this series is $(1-x)^{-1}$, *if x is numerically less than unity.*

Let us now apply the binomial rule, *shewn* to hold for a positive integral exponent. This would give

$$1+(-1)(-x)+\frac{(-1)(-2)}{1.2}(-x)^2+\ldots+\frac{(-1)(-2)\ldots(-1-r+1)}{1.2.3\ldots r}(-x)^r+\ldots$$

or when reduced

$$1+x+x^2+x^3+\ldots\ldots$$

a series which will not terminate. The result is in agreement with the expansion found by division.

Ex. 2. Examine whether an expansion in ascending powers of x can be found for $(1-x)^{-2}$.

We have

$$(1-x)^{-2}=\frac{1}{(1-x)^2}=\frac{1}{1-2x+x^2}.$$

Let the division be performed :

$$\begin{array}{l}
1-2x+x^2)\ 1 \qquad\qquad (1+2x+3x^2+\\
\qquad\qquad\ \ \underline{1-2x+x^2}\\
\qquad\qquad\qquad 2x-x^2\\
\qquad\qquad\qquad \underline{2x-4x^2+2x^3}\\
\qquad\qquad\qquad\qquad 3x^2-2x^3\\
\qquad\qquad\qquad\qquad \underline{3x^2-6x^3+3x^4}\\
\qquad\qquad\qquad\qquad\qquad 4x^3-3x^4
\end{array}$$

It is seen that the division will not terminate and it may also be seen that the law of terms in the quotient will continue. This last, however, is more easily seen if we note that

$$(1-x)^{-2}=(1-x)^{-1}\div(1-x),$$

and assume that it is permissible to divide an infinite series. We then have

$$(1-x)^{-2}=(1+x+x^2+\ldots\ldots \textit{ in inf.})\div(1-x).$$

Let the division be undertaken :

$$\begin{array}{l}
1-x)\ 1+x+x^2+x^3+\ldots\ldots\ (1+2x+3x^2+\ldots\ldots\\
\qquad\ \ \underline{1-x}\\
\qquad\qquad 2x+\ x^2\\
\qquad\qquad \underline{2x-2x^2}\\
\qquad\qquad\qquad 3x^2+x^3\\
\qquad\qquad\qquad\qquad \ldots\ldots
\end{array}$$

The law of terms in the quotient is now evident.

Now, let us apply the binomial rule to $(1-x)^{-2}$. This would give

$$1+(-2)(-x)+\frac{(-2)(-3)}{1.2}(-x)^2+\ldots+\frac{(-2)(-3)\ldots(-3-r+1)}{1.2\ \ldots\ r}(-x)^r+\ldots$$

which reduces to

$$1+2x+3x^2+\ldots\ldots+(r+1)x^r+\ldots\ldots$$

The result is in agreement with what was previously seen.

The application of the rule here as in the earlier example furnishes an infinite series.

Referring to Ex. 4, p. 85, we see that we can find the sum of n terms of the series

$$1+2x+3x^2+\ldots\ldots$$

and *if x is numerically less than unity*, it can be shewn that the limit of the infinite series is $(1-x)^{-2}$.

Ex. 3. Find an expression in ascending powers of x for $(1+x)^{\frac{1}{2}}$.

We have $(1+x)^{\frac{1}{2}}=\surd(1+x)$. Let the square root be extracted,

$$\begin{array}{ll}
 & 1+x\,\Big(\,1+\frac{x}{2}-\frac{x^2}{8} \\
 & 1 \\
\cline{2-2}
2+\frac{x}{2} & \quad +x \\
 & \quad +x+\frac{x^2}{4} \\
\cline{2-2}
2+x-\frac{x^2}{8} & \qquad -\frac{x^2}{4} \\
 & \qquad -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\cline{2-2}
 & \qquad\qquad \ldots\ldots
\end{array}$$

If the binomial rule were applied we should obtain

$$1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x^2+\ldots\ldots$$

or

$$1+\frac{x}{2}-\frac{x^2}{8}+\ldots\ldots$$

a result which agrees with the value found for $(1+x)^{\frac{1}{2}}$. It is plain that the series will not terminate, and it could be shewn that the infinite series has a meaning only *when x is numerically less than unity.*

It is also to be noted that $(1+x)^{\frac{1}{2}}$ has two values. In finding the square root we might have started with -1 as well as with $+1$ and the two values found would differ only in sign. The binomial rule gives only one of the roots.

A study of the preceding examples would seem to lead to the conclusion that the Binomial Theorem is valid, under certain restrictions, for all integral or fractional positive or negative exponents. The student who is not making a special study of mathematics may, for the case of a fractional or negative exponent, assume its truth as thus stated:

If

(1) *n is any positive integer and x any number,*

or if

(2) *n is a positive fraction or a negative integer or fraction while x is numerically less than unity,*

then

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \ldots\ldots + \frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}x^r + \ldots\ldots$$

It is to be noted that if n is a positive integer the series terminates through the appearance of a factor zero in the numerator of a term; if n is not a positive integer no such factor can appear and the series will not terminate.

If it is a question of the expansion of $(a+x)^n$ where n is fractional or negative, it is well, as a rule, to regard this as $a^n\left(1+\frac{x}{a}\right)^n$.

In Art. 5 of this chapter will be given a proof of this theorem which may, if desired, be now studied. The following examples, however, do not depend upon the *method of proof* but only on the *result* which as said may be assumed.

Ex. 1. Expand $(1-x)^{-3}$.

It follows from the general theorem that

$$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{1.2}(-x)^2 + \ldots\ldots$$

The general term is

$$\frac{(-3)(-4)\ldots\ldots(-3-r+1)}{1.2\ldots\ldots r}(-x)^r.$$

This equals

$$(-1)^r.\frac{3.4\ldots\ldots(r+2)}{1.2\ldots\ldots r}(-x)^r, \text{ or } \frac{(r+1)(r+2)}{1.2}x^r.$$

$$\therefore\ (1-x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \ldots\ldots + \frac{(r+1)(r+2)}{1.2}x^r + \ldots\ldots$$

Ex. 2. Expand $(a+x)^{-2}$

$$(a+x)^{-2}=a^{-2}.\left(1+\frac{x}{a}\right)^{-2}=\frac{1}{a^2}\cdot\left\{1+(-2)\frac{x}{a}+\frac{(-2)(-3)}{1.2}\cdot\left(\frac{x}{a}\right)^2+\ldots\ldots\right\}$$

The general term of the expansion within the brackets is

$$\frac{(-2)(-3)\ldots\ldots(-2-r+1)}{1.2\ldots\ldots r}\cdot\left(\frac{x}{a}\right)^r \text{ or } (-1)^r.\ (r+1)\frac{x^r}{a^r}$$

$$\therefore\ (a+x)^{-2}=\frac{1}{a^2}-2\frac{x}{a^3}+3\frac{x^2}{a^4}-\ldots\ldots+(-1)^r(r+1).\frac{x^r}{a^{r+2}}+\ldots\ldots$$

Ex. 3. Find the coefficient of x^r in the expansion of $\left(\frac{1+x}{1-x}\right)^2$.

$$\left(\frac{1+x}{1-x}\right)^2=(1+x)^2\,(1-x)^{-2}=(1+2x+x^2)\,(1+2x+3x^2+\ldots\ldots+\overline{r+1}\,x^r+\ldots\ldots)$$

In this last product the coefficient of x^r is seen to be

$$(r+1)+2.r+(r-1) \text{ or } 4r.$$

This result may be obtained otherwise in noting that

$$\left(\frac{1+x}{1-x}\right)^2=\frac{(1-x)^2+4x}{(1-x)^2}=1+4x\,(1-x)^{-2}.$$

Ex. 4. Find the sum of the first $r+1$ coefficients in the expansion of $(1-x)^n$ where n may be any positive or negative integer or fraction.

Let

$$(1-x)^n\ =c_0+c_1x+c_2x^2+\ldots\ldots+c_rx^r+\ldots\ldots$$

Also we have

$$(1-x)^{-1}=1+\ x+\ x^2+\ldots\ldots+\ x^r+\ldots\ldots$$

Then $c_0+c_1+\ldots\ldots+c_r$ equals the coefficient of x^r in the expansion of the product $(1-x)^n(1-x)^{-1}$ or $(1-x)^{n-1}$. This coefficient equals

$$\frac{(n-1)(n-2)\ldots\ldots(n-1-r+1)}{1.2\ldots\ldots\ldots r} \text{ or } \frac{(n-1)(n-2)\ldots\ldots(n-r)}{r\,!}.$$

EXERCISES

1. The following special expansions are so important that the student should be familiar with them. They are put as exercises and the general term in each case is to be found.

(1) $(1-x)^{-1}=1+x+x^2+x^3+x^4+\dots\dots$

(2) $(1+x)^{-1}=1-x+x^2-x^3+x^4-\dots\dots$

(3) $(1-x)^{-2}=1+2x+3x^2+4x^3+5x^4+\dots\dots$

(4) $(1-x)^{-3}=1+3x+\frac{3.4}{1.2}x^2+\frac{4.5}{1.2}x^3+\frac{5.6}{1.2}x^4+\dots\dots$

(5) $(1-x)^{-4}=1+4x+\frac{4.5}{1.2}x^2+\frac{4.5.6}{1.2.3}x^3+\frac{5.6.7}{1.2.3}x^4+\dots\dots$

(6) $(1-x)^{-m}=1+mx+\frac{m(m+1)}{1.2}x^2+\frac{m(m+1)(m+2)}{1.2.3}x^3+\dots\dots$

2. Find the general term in the expansion of each of the following :

$$(1-x)^{\frac{1}{2}},\ (1-x)^{-\frac{1}{2}},\ (a+x)^{\frac{3}{2}},\ (2a-3x)^{-5}.$$

3. Find the first negative term in the expansion of $(1+x)^{\frac{9}{4}}$.

4. Expand to 5 terms in a series of powers of the fractions appearing in the binomial :

$$\left(1-\frac{1}{10}\right)^{\frac{1}{2}},\ \left(1+\frac{1}{3}\right)^{-3},\ \left(1+\frac{1}{2}\right)^{-\frac{1}{3}}.$$

5. Find the general term in the expansion of each of the following :

$$(1+x)^{\frac{1}{3}},\ (1+3x)^{-\frac{2}{3}},\ (1-x)^{-\frac{1}{5}}.$$

6. Find the coefficient of x^r in the expansion of each of the following :

$$\frac{1+x}{(1-x)^3},\ \frac{(1+x)^2}{(1-x)^3},\ \frac{(1+x)^3}{(1-x)^2}.$$

7. Expand to four terms each of the following :

$$(1-x)^{\frac{1}{2}}\ (1-x)^{\frac{1}{3}},\ (1+x)^{\frac{1}{3}}\ (1+x)^{\frac{1}{2}}\ (1+2x)^{\frac{2}{3}}.$$

8. Find the sum of the first $r+1$ coefficients of the expansion of $\frac{1-x}{(1+x)^m}$.

9. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\frac{(2r)\,!}{(r\,!)^2}$

10. Recover the power of a binomial which leads to each of the following:

(1) $1+2\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)^2+4\left(\frac{1}{3}\right)^3+\ldots\ldots$

(2) $1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{2^3}+\ldots\ldots$

(3) $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\ldots\ldots$

(4) $1+\frac{1}{5}+\frac{1.4}{1.2}\cdot\frac{1}{5^2}+\frac{1.4.7}{1.2.3}\cdot\frac{1}{5^3}+\ldots\ldots$

4. **Approximations.** When in a problem the result is given in the form of an infinite series it is in general necessary, for practical purposes, to accept the **approximation** furnished by some definite and small number of terms. The following examples will illustrate this fact. The question of the degree of the approximation will not be here investigated, the student being referred to the excellent Algebra of Chrystal, Vol. II, p. 192, *et seq.*

Ex. 1. A cube of copper of edge 1 in. at 0°C. is brought to a temperature of 1°C. It is found that each edge has been increased in length 0·000017 in.; find the increase in volume. The volume at 1°C. is $(1+0{\cdot}000017)^3$ cubic in. and this equals

$$(1+3\times0{\cdot}000017+3\times0{\cdot}000017^2+0{\cdot}000017^3)\text{ c. in.}$$

The last two terms are very small compared with the second term and the volume to a degree of approximation that makes it *practically correct* is $(1+3\times0{\cdot}000017)$ c. in., so that the expansion in volume is $3\times0{\cdot}000017$ or 0·000051 c. in.

Here 0·000017 is the coefficient of linear expansion of copper and 0·000051 the coefficient of cubical expansion.

In like manner if the coefficient of linear expansion of a given metal is x the coefficient of cubical expansion is $3x$.

Ex. 2. Find approximately the square root of 99.

$$\sqrt{99}=(100-1)^{\frac{1}{2}}=100^{\frac{1}{2}}\left(1-\frac{1}{100}\right)^{\frac{1}{2}}=10\left(1-\frac{1}{100}\right)^{\frac{1}{2}}$$

$$=10\left(1-\frac{1}{2}\cdot\frac{1}{100}-\frac{1}{8}\cdot\frac{1}{100^2}-\frac{1}{16}\cdot\frac{1}{100^3}-\ldots\ldots\right)$$

The terms within the brackets become small very rapidly, and if we take the first three terms as an approximation, the result is 9·949875, which is correct to 5 places of decimals, the root having the digit 4 in the sixth place. By taking four terms we get a result correct to eight places of decimals.

Ex. 3. If x is a quantity so small that the cubes and higher powers of x may be neglected, find the approximation to the value of

$$\frac{(1+2x)^{\frac{1}{2}}\,(1-x)^{\frac{3}{2}}}{(2+3x)^2}.$$

$$\text{This fraction } = \frac{(1+2x)^{\frac{1}{2}}\,(1-x)^{\frac{3}{2}}\left(1+\frac{3x}{2}\right)^{-2}}{4}$$

$$= \frac{\left(1+x-\frac{x^2}{2}\right)\left(1-\frac{3x}{2}+\frac{3x^2}{8}\right)\left(1-3x-\frac{27x^2}{4}\right)}{4}, \text{ approximately}$$

$$= \frac{\left(1-\frac{1}{2}x+\frac{13}{8}x^2\right)\left(1-3x-\frac{27x^2}{4}\right)}{4}, \quad \text{"}$$

$$= \frac{1-\frac{7x}{2}-\frac{29x^2}{8}}{4}, \quad \text{"}$$

$$= \frac{1}{4}-\frac{7x}{8}-\frac{29x^2}{32}, \text{ neglecting powers higher than the third.}$$

EXERCISES

1. Find approximately the square root of each of the following numbers:

24, 80, 620, 224, 1220.

2. Find approximately the value of each of the following:

$63^{\frac{1}{3}}$, $97^{\frac{1}{2}}$, $1 \div 47^{\frac{1}{2}}$, $240^{\frac{1}{5}}$ $1 \div 127^{\frac{1}{3}}$.

3. If the coefficient of cubical expansion of a certain metal is 0 000078, find the coefficient of linear expansion.

4. If x is so small that its second and higher powers may be neglected find the value of each of the following:

$$(1+2x)^{\frac{1}{2}}(1-3x)^{\frac{2}{3}},\ (4+x)^{-\frac{1}{2}}(1-x)^{-3},\ (8+x)^{-\frac{1}{3}}(1+x)^{\frac{2}{7}},$$

$$\frac{(1+2x)^{\frac{1}{2}}+(1-3x)^{-\frac{2}{3}}}{(1+4x)^{\frac{3}{4}}+(1+5x)^{\frac{1}{5}}},\ \frac{(5+9x)^{-1}+(3+2x)^{-2}}{(1+x)^{\frac{1}{3}}}.$$

5. Proof of the Binomial Theorem *for a Positive Fractional Exponent and for a Negative Integral or Fractional Exponent.*

The following proof of the theorem, under stated assumptions, for the case in which the exponent is fractional or negative, is now added.

The series

$$1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+.\quad ..+\frac{m(m-1)\ldots\ldots(m-r+1)}{1.2\ldots\ldots r}x^r+\ldots\ldots$$

will terminate if m is a positive integer and its value is then $(1+x)^m$. If m is not a positive integer no factor 0 can be introduced into the numerator of any term so that the series will not terminate; this infinite series has a finite limit when $|\,x\,|<1$, a fact which will be *assumed*, and under this supposition as to the values of m and x, it is proposed to find the value of the series. The value of the series depends on that of m; it is then a certain function of m which will be denoted by $f(m)$. Thus if m is a positive integer $f(m)=(1+x)^m$. Then if $\binom{m}{r}$ is written for $\dfrac{m(m-1)\ldots\ldots(m-r+1)}{1.2\ldots\ldots r}$ whatever be the value of m,

$$f(m)=1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots\ldots+\binom{m}{r}x^r+\ldots\ldots$$

Also, if m is replaced by n,

$$f(n)=1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots\ldots+\binom{n}{r}x^r+\ldots\ldots$$

Assume now that it is permissible to multiply two infinite series with finite limit as if they were polynomials; then

$$f(m)\,.\,f(n)=1+\left\{\binom{m}{1}+\binom{n}{1}\right\}x+\text{an infinite series of higher powers of }x.$$

In this product, the coefficient of x^r is

$$\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\ldots\ldots\ \binom{m}{r}\binom{n}{r}.$$

This series of fractions involving m and n may be added to form one fraction and the way in which m and n appear in the result will be the same whatever values m and n have. But when m and n are positive integers (See *Ex. 3*, p. 102), the sum is

$$\binom{m+n}{r}; \text{ or } \frac{(m+n)\,(m+n-1)\,\ldots\ldots\,(m+n-r+1)}{1.2\,\ldots\ldots\,r}$$

which must then be the sum whatever be m and n. Therefore, whatever be the values of m and n,

$$f(m)\,.\,f(n) = 1 + (m+n)\,x + \frac{(m+n)\,(m+n-1)}{1.2}\,x^2 + \ldots\ldots$$

$$= 1 + \binom{m+n}{1}\,x + \binom{m+n}{2}\,x^2 + \ldots\ldots + \binom{m+n}{r}\,x^r + \ldots\ldots$$

$\therefore\ f(m)\,.\,f(n) = f(m+n).$

Hence, also,

$$f(m)\,.\,f(n)\,.\,f(p) = f(m)\,.\,f(n+p) = f(m+n+p),$$

and generally,

$$f(m)\,.\,f(n)\,\ldots\ldots\,f(t) = f(m+n+\ldots\ldots+t).$$

Suppose that here there are q factors and put $m, n, \ldots\ldots, t$ each equal to $\frac{p}{q}$ where p and q are positive integers. Then

$$\left\{f\left(\frac{p}{q}\right)\right\}^r = f\left(\frac{p}{q} \times q\right) = f(p)$$

$$= (1+x)^p, \text{ since } p \text{ is a positive integer.}$$

$$\therefore\ f\left(\frac{p}{q}\right) = (1+x)^{\frac{p}{q}}.$$

Hence, if $f\left(\frac{p}{q}\right)$ is written at length,

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1.2}x^2 + \ldots + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\ldots\left(\frac{p}{q}-r+1\right)}{1.2\,\ldots\ldots\ldots\,r}x^r + \ldots$$

and the binomial rule applies to the case in which the exponent is a positive fraction.

As there are more than one qth root (q indeed) of which one or at most two are real, it is necessary to determine which qth root is given by the expansion. Let x grow to the value 0; then $(1+x)^{\frac{p}{q}}$ becomes $1^{\frac{p}{q}}$ and the series becomes 1; hence, for these to be equal, the qth root taken must be the arithmetical root.

Next let m be a negative integer or fraction, and put it equal to $-m'$ where m' is positive. Then

$$f(-m').f(m')=f(-m'+m')=f(0)=1$$

$$\therefore \quad f(-m')=\frac{1}{f(m')}.$$

But m' being a positive integer or fraction $f(m')=(1+x)^{m'}$

$$\therefore \quad f(-m')=\frac{1}{(1+x)^{m'}}=(1+x)^{-m'}$$

$$\therefore \quad (1+x)^m=f(m),$$

or if $f(m)$ is written at length

$$(1+x)^m=1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\ldots\ldots+\frac{m(m-1)\ldots\ldots(m-r+1)}{1.2\ldots\ldots r}x^r+\ldots\ldots$$

Hence the binomial rule applies to the case in which m is a negative integer or fraction.

6. The number of Homogeneous Products. Let $a, b, c, \ldots\ldots, h$, be n given letters, and consider the product of the n infinite series:

$$\begin{array}{l} 1+ax+a^2x^2+\ldots\ldots+a^rx^r+\ldots\ldots \\ 1+bx+b^2x^2+\ldots\ldots+b^rx^r+\ldots\ldots \\ 1+cx+c^2x^2+\ldots\ldots+c^rx^r+\ldots\ldots \\ \qquad\qquad\quad \ldots\ldots \\ 1+hx+h^2x^2+\ldots\ldots+h^rx^r+\ldots\ldots \end{array}$$

In this product the coefficient of x^r will be the sum of all possible terms of r dimensions that can be made from the n given letters and repetitions of them, *i.e.*, will be the sum of the homogeneous products of n things r at a time. If each of $a, b, c, \ldots\ldots, h$ be put equal to 1, each term in this sum is 1, and the sum is the *number* of such products. But when the letters are thus replaced by unity, the product of infinite series becomes $(1+x+x^2+\ldots\ldots)^n$ or $(1-x)^{-n}$, and in this the coefficient of x^r is

$$\frac{n(n+1)\ldots\ldots(n+r-1)}{1.2.\ldots\ldots r},$$

and this is therefore the *number* of homogeneous products of n things r at a time.

7. The Numerically Greatest Term. It is proposed to find the number of the term or terms of greatest numerical value in the expansion of $(1+x)^n$. The $(r+1)$th term is formed from the rth by multiplying by

$$\frac{n-r+1}{r}.x \text{ or } \left(\frac{n+1}{r}-1\right)x$$

which may be called the *multiplier for the* $(r+1)$*th term.* As we are concerned only with the numerical value of terms, we need consider only the numerical value of this multiplier, and may then suppose x to be positive.

I. Suppose n a positive integer.

Then since r cannot be greater than $n+1$ the multiplier is always positive. Now

$$\text{the } (r+1)\text{th term} >=< \text{the } r\text{th term}$$

$$\text{as } \left(\frac{n+1}{r}-1\right)x >=< 1$$

$$\therefore \text{ as } \frac{n+1}{r} >=< \frac{1}{x}+1$$

$$\therefore \text{ ,, } \frac{r}{n+1} >=< \frac{x}{x+1}$$

$$\therefore \text{ ,, } r >=< \frac{(n+1)x}{x+1}.$$

As long as r is less than this value the terms continue to increase and, when r passes this value, to diminish. If $\frac{(n+1)x}{x+1}$ is an integer p, the multiplier for $r=p$ is 1, and the pth and $(p+1)$th terms are equal and greater than any other term. If $\frac{(n+1)x}{x+1}$ is fractional and between the integers p and $p+1$, then $r=p$ is the last value that makes the multiplier for the $(r+1)$th term greater than unity and the $(p+1)$th term is the greatest.

II. Suppose n a positive fraction, or a negative integer or fraction. We have then to take x less than 1.

(1) Let n be a positive or negative proper fraction.

Then $$0<n+1<2$$

$$\therefore \quad 0<\frac{n+1}{r}<2, \text{ for all values of } r$$

$$\therefore \quad -1<\frac{n+1}{r}-1<1, \text{ for all values of } r$$

Thus, x being less than 1, $\left(\frac{n+1}{r}-1\right)x$ is numerically less than 1, so that the multiplier for the $(r+1)$th term is always numerically less than 1 and the first term is the greatest.

(2) Let $n=-1$.

The expansion is then $1+x+x^2+\ldots\ldots$, and the first term is the greatest.

(3) Let $n<-1$.

Put $n=-m$ where m is positive and greater than 1. The multiplier is then numerically $\left(\frac{m+r-1}{r}\right)x$ or $\left(\frac{m-1}{r}+1\right)x$. Then

$$\text{the } (r+1)\text{th term} > = < \text{the } r\text{th term}$$

$$\text{as } \left(\frac{m-1}{r}+1\right)x > = < 1$$

$$\therefore \quad \text{"} \quad \frac{m-1}{r} > = < \frac{1}{x}-1$$

$$\therefore \quad \text{"} \quad \frac{r}{m-1} > = < \frac{x}{1-x}$$

$$\therefore \quad \text{"} \quad r > = < \frac{(m-1)x}{1-x}, \text{ or } \frac{-(n+1)x}{1-x}$$

Therefore as in I, if this last value is an integer p, the pth and $(p+1)$th terms are equal and greater than any other term, while, if it lies between the integers p and $p+1$, the $(p+1)$th term is the greatest term.

(4) Let $n>1$.

When $r>n+1$ it is plain that the multiplier $\left(\frac{n+1}{r}-1\right)x$ is numerically less than unity, so that the greatest term is among those for $r<n+1$, since n not being an integer, r cannot equal $n+1$. The multiplier is in this case positive. As before we find that

$$(r+1)\text{th term} > = < r\text{th term},$$

as
$$r > = < \frac{(n+1)x}{1+x},$$

and that, if this value is an integer p, the pth and $(p+1)$th terms are equal, and are the greatest terms, while if this value lies between the integers p and $p+1$, the $(p+1)$th term is the greatest.

EXERCISES

1. Find the number of the greatest term in each of the following expansions, reproducing the reasoning of this article:

(1) $(1-x)^{-3}$ for $x=\frac{3}{4}$.

(2) $(1+x)^{-\frac{2}{3}}$ for $x=\frac{5}{7}$.

(3) $(1+x)^{9}$ for $x=\frac{1}{2}$ and for $x=\frac{3}{2}$.

(4) $(1+x)^{\frac{13}{2}}$ for $x=\frac{2}{3}$.

In each case write down the multiplier for the 2nd, 3rd, ... terms until the greatest term is reached.

2. Find the number of the greatest term in the expansion of each of the following:

(1) $(1+x)^{-2}$ for $x=\frac{5}{6}$.

(2) $(1+x)^{13}$ for $x=\frac{5}{4}$.

(3) $(1+x)^{-7}$ for $x=\frac{4}{5}$.

(4) $(1+x)^{\frac{37}{3}}$ for $x=\frac{8}{9}$.

EXAMPLES

(NOTE.—When, in the following examples, the letters $c_0, c_1, c_2, \ldots\ldots$ appear, they are supposed to be the coefficients in the expansion of $(1+x)^n$ where n is a positive integer, and the last of them will therefore be c_n.)

1. Find the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{2n}$.

2. If m and n are positive integers, shew that the coefficient of x^m in the expansion of $(1-x)^{-(n+1)}$ is equal to the coefficient of x^n in the expansion of $(1-x)^{-(m+1)}$.

3. Shew that, in the infinite series which gives a binomial expansion, the terms are sooner or later of the same sign or of alternate signs.

Illustrate each possibility.

4. Shew that

$$\frac{c_1}{c_0}+2\frac{c_2}{c_1}+3\frac{c_3}{c_2}+\ldots\ldots+n\frac{c_n}{c_{n-1}}=\frac{n(n+1)}{2}.$$

5. Find the coefficient of x^{n+r} in the expansions of $\frac{(1+x)^n}{1-x}$ and $\frac{(1+x)^n}{(1-x)^2}$.

6. Sum to infinity:

(1) $1+\frac{1}{10}+\frac{1.3}{10.20}+\frac{1.3.5}{10.20.30}+\ldots\ldots$

(2) $\frac{1}{9}+\frac{1.4}{9.18}+\frac{1.4.7}{9.18.27}+\ldots\ldots$

7. Find the sum of the first $n+r$ coefficients in the expansion of $\frac{(1+x)^n}{1-x}$.

8. If n is a positive integer, shew that

$$1 - \frac{n^2}{1^2} + \frac{n^2(n^2 - 1^2)}{1^2 . 2^2} - \ldots\ldots = 0.$$

How many terms are there in this series?

9. Shew that

$$(c_0 + c_1)(c_1 + c_2) \ldots\ldots (c_{n-1} + c_n) = c_0 c_1 \ldots\ldots c_n . \frac{(n+1)^n}{n\,!}.$$

10. Find the value of the remainder after n terms in the expansions of $(1 - x)^{-1}$ and $(1 - x)^{-2}$.

11. Shew that

$$c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \ldots\ldots + \frac{c_n}{n+1} = \frac{2^{n+1} - 1}{n+1}.$$

12. Find the coefficient of $x^5 y^7 z^{13}$ in the expansion of $(x + y + z)^{25}$.

13. Shew that

$$c_1 - 2c_2 + 3c_3 - \ldots\ldots + n(-1)^{n-1} c_n = 0.$$

14. By treating $1 - 2x + 3x^2$ as a binomial find the coefficient of x^4 in the expansion of $(1 - 2x + 3x^2)^n$.

15. Shew that

$$c_0 + 2c_1 + 3c_2 + \ldots\ldots + (n+1)c_n = 2^n + n 2^{n-1}.$$

16. Shew that

$$c_1^2 + 2c_2^2 + \ldots\ldots + nc_n^2 = (2n - 1)\,! \div (n - 1)\,!\ (n - 1)\,!.$$

17. If n is an odd positive integer shew that the integral part of $(\sqrt{2} + 1)^n$ is $(\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n$.

18. Shew that

$$c_0^2 - c_1^2 + \ldots\ldots + (-1)^n c_n^2$$

is equal to 0 if n is odd and to $(-1)^{\frac{n}{2}}\, n\,! \div (\tfrac{1}{2}n)\,!\ (\tfrac{1}{2}n)\,!$ if n is even.

19. If n is a positive integer shew that the integral part of $(2 + \sqrt{3})^n$ is an odd integer.

CHAPTER VII

THE EXPONENTIAL AND LOGARITHMIC SERIES

1. The Exponential Series. It is proposed to make a brief study of the infinite series

$$1+\frac{x}{1}+\frac{x^2}{1.2}+\ldots\ldots+\frac{x^r}{1.2\ldots\ldots r}+\ldots\ldots \qquad (1)$$

a series which, on account of its simple form, might very easily have suggested itself for examination. This series has a finite limit for all finite values of x, a fact which will be assumed. Thus for $x=1$ the series is

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots\ldots+\frac{1}{1.2.3\ldots\ldots r}+\ldots\ldots \qquad (2)$$

This last series is seen to be less than

$$1+1+\frac{1}{2}+\frac{1}{2.2}+\frac{1}{2.2.2}+\ldots\ldots+\frac{1}{2^{r-1}}+\ldots\ldots$$

which after the first term is an infinite geometrical progression with common ratio $\frac{1}{2}$ so that its sum is

$$1+\frac{1}{1-\frac{1}{2}} \text{ or } 3.$$

Thus the series (2) has a finite limit between 2 and 3. This limit can, by taking a sufficient number of terms, be found to any degree of accuracy but it cannot be computed exactly. Its value is denoted by e so that

$$e=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\ldots \textit{ in inf.} \qquad (I)$$

Approximately $e=2{\cdot}7182818$

Denote the series (1) by $F(x)$, thus indicating that it is a function of x. Then putting for x the values m and n we have

$$F(m) = 1 + \frac{m}{1\,!} + \frac{m}{2\,!} + \ldots\ldots + \frac{m}{r\,!} + \ldots\ldots$$

$$F(n) = 1 + \frac{n}{1\,!} + \frac{n}{2\,!} + \ldots\ldots + \frac{n}{r\,!} + \ldots\ldots$$

Assuming that these series may be multiplied as if they were two polynomials we have

$$F(m) \, . \, F(n) = 1 + \frac{1}{1\,!}(m+n) + \text{terms of higher dimensions in } m \text{ and } n.$$

In this product the term of r dimensions in m and n is

$$\frac{m^r}{r\,!} + \frac{m^{r-1}}{(r-1)\,!} \cdot \frac{n}{1!} + \frac{m^{r-2}}{(r-2)\,!} \cdot \frac{n^2}{2!} + \ldots\ldots + \frac{n^r}{r!}$$

which can be put in the form

$$\frac{1}{r!}\left[m^r + \frac{r}{1}\,m^{r-1}n + \frac{r(r-1)}{1.2}\,m^{r-2}n^2 + \ldots\ldots + n^r\right]$$

which is equal to $$\frac{(m+n)}{r\,!}.$$

Then giving to r the values 1, 2, 3, ... we have

$$F(m) \, . \, F(n) = 1 + \frac{(m+n)}{1\,!} + \frac{(m+n)^2}{2\,!} + \ldots\ldots + \frac{(m+n)^r}{r\,!} + \ldots\ldots$$

that is, the product is the result of putting $m+n$ in place of (x) in (I). Therefore,

$$F(m) \, . \, F(n) = F(m+n). \qquad (II)$$

Let, now, x be a positive integer. Then, by repeated application of (II), it follows that

$$F(1) \, . \, F(1) \ldots\ldots \text{ to } x \text{ factors} = F(1 + 1 + \ldots\ldots \text{ to } x \text{ terms}).$$

$$\therefore \ \left\{F(1)\right\}^x = F(x) \qquad (III)$$

Next let $\frac{p}{q}$ be any positive fraction, p and q being integers. Then

$$F\left(\frac{p}{q}\right). \, F\left(\frac{p}{q}\right) \ldots\ldots \text{ to } q \text{ factors} = F\left(\frac{p}{q} + \frac{p}{q} + \ldots\ldots \text{ to } q \text{ terms}\right)$$

$\therefore \ \left\{F\left(\frac{p}{q}\right)\right\}^q = F(p) = \left\{F(1)\right\}^p$, by (III) since p is a positive integer.

Then extracting the qth root

$$F\left(\frac{p}{q}\right)=\left\{F(1)\right\}^{\frac{p}{q}} \qquad (IV)$$

Thus from (III) and (IV) it follows that for all positive integral or fractional values of x

$$\left\{F(1)\right\}^{x}=F(x).$$

Finally let x be a negative fraction or integer and equal to $-y$ so that y is positive. By (II) we have

$$F(-y).F(y)=F(-y+y)=F(0) \text{ which is seen to be } 1.$$

$$\therefore \ F(-y)=\frac{1}{F(y)}=\frac{1}{\left\{F(1)\right\}^{y}}=\left\{F(1)\right\}^{-y}$$

Thus also, replacing $-y$ by x, we have when x is a negative integer or fraction

$$F(x)=\left\{F(1)\right\}^{x}$$

Hence for all positive or negative integral or fractional values of x,

$$\left\{F(1)\right\}^{x}=F(x).$$

But $F(1)=e(=2{\cdot}7182818$ approximately$)$. Therefore

$$e^{x}=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots+\frac{x^r}{r!}+\ldots\ldots \qquad (V)$$

This is the **exponential theorem.**

2. **Logarithmic Theorem.** In (V) of the preceding article replace x by $x\log_e a$. Then, since $e^{x\log_e a}=\left\{e^{\log_e a}\right\}^{x}=a^{x}$ by the definition of logarithm, it follows that

$$a^{x}=1+\frac{x\log_e a}{1}+\frac{x^2.(\log_e a)^2}{1.2}+\ldots\ldots$$

In this put $x=1+y$.

$$\therefore \quad (1+y)^{x}=1+\frac{x\log_e(1+y)}{1}+\frac{x^2.\left\{\log_e(1+y)\right\}^2}{1.2}+\ldots\ldots$$

Let y be numerically less than 1 so that we may expand $(1+y)^x$ by the binomial theorem. Then

$$(1+y)^x = 1 + x.y + \frac{x(x-1)}{1.2}y^2 + \ldots + \frac{x(x-1)\ldots(x-r+1)}{1.2\ldots r}y^r + \ldots$$

Then equating the coefficient of x in the two values of $(1+y)^x$, we have

$$\log_e(1+y) = \frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots.$$

This is valid for values of y numerically less than 1. Replacing y by x as the theorem is generally quoted in terms of x, we have for $|x|<1$

$$\log_e(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots..$$

This is the **logarithmic theorem.**

It can be shewn to be valid for $x=1$, though not for $x=1$. By giving to x the value $-\frac{1}{2}$ we can find to any degree of accuracy $\log_e 2$ and *then* by putting $x = -\frac{1}{3}$ the value of $\log_e 3$ and so on. This is not the best way of finding these logarithms, as the series may be modified to give the approximate values more readily, but this matter we shall not deal with further.

When the logarithms of successive integers to the base e have been found they can be changed to logarithms to base 10. For having found $\log_e 10$ we have for any number N

$$\log_{10}\text{N} = \frac{\log_e \text{N}}{\log_e 10}.$$

ANSWERS

EXERCISES : PAGE 5

1. (1) $x(x+1)-x^2=13$; (2) $3x+15=2(x+15)$; (3) $2(100-x\times 100\div 10)=100-x\times 100\div 11$. 2. (1) A property of every three consecutive integers; (2) A property for every division of the line.

EXERCISES : PAGE 6

1. (1) 11; (2) 12; (3) pqr; (4) abc. 2. 15 ft., 27 ft.

EXERCISES : PAGE 9

1. $0, x=0$; $-1, x=0$; $5\frac{3}{4}, x=\frac{1}{2}$; $2, x=1$. 2. $-5\frac{3}{4}, x=\frac{1}{2}$; $0, x=0$; $1, x=0$; $4, x=1$. 3. $-6\frac{1}{8}, x=\frac{3}{4}$, min.; $4\frac{11}{12}, x=\frac{5}{6}$, min.; $7\frac{1}{8}, x=1\frac{1}{4}$, max.

EXERCISES : PAGE 15

1. $\left\{x+(7+\sqrt{69})\div 2\right\}\left\{x+(7-\sqrt{69})\div 2\right\}$; $(x-1)(2x-5)$; $\left\{2x+(\sqrt{73}+5)\div 4\right\}\left\{(\sqrt{73}-5)\div 4-2x\right\}$; $7\left\{x-(11+\sqrt{-19})\div 14\right\}\left\{x-(11-\sqrt{-19})\div 14\right\}$. 2. $(7\pm\sqrt{37})\div 2$; $(2\pm\sqrt{19})\div 3$; $(-3\pm\sqrt{149})\div 10$; $(11\pm\sqrt{131})\div 6$. 3. Real and positive; real, one positive and one negative, the former the greater numerically; real and negative; real and positive. 4. $x^2-9x+11=0$. 5. $ax^2+(b-2ah)x+c-bh+ah^2=0$. 6. $x^2-10x+12=0$. 7. $x^2+mpx+m^2q=0$. 9. (1) $4x^2-37x+9=0$; (2) $3x^2-7x+2=0$; (3) $8x^2-42x+27=0$. 10. (1) $a^2x^2-(b^2-2ac)x+c^2=0$; (2) $cx^2+bx+a=0$; (3) $c^2x^2-(b^2-2ac)x+a^2=0$; (4) $a^3x^2+b(b^2-3ac)x+c^3=0$; (5) $a^3x^2+abcx+c^3=0$; (6) $ac^2-(b^2-4ac)x+1=0$. 11. $(q-s)^2=(p-r)(rq-ps)$. 12. $a\div p=b\div q=c\div r$. 14. Zero for $x=4$ or -3; negative for x between 4 and -3; positive for other values. 15. Always positive. 16. $a(\sqrt{5}-1)\div 2$ and $a(3-\sqrt{5})\div 2$; $a(3+\sqrt{5})\div 2$ and $a(-1-\sqrt{5})\div 2$.

EXERCISES : PAGE 21

1. $\pm\sqrt{6}\div 2$, $\pm\sqrt{35}\div 5$. 2. $\frac{7}{5}, \frac{5}{7}, (-1\pm\sqrt{-3})\div 2$. 3. $4, -13, (9\pm 3\sqrt{-31})\div 2$. 4. $5, -\frac{5}{2}, \left\{(5\pm\sqrt{137})\div 4\right\}$. 5. $2, 2, (9\pm\sqrt{337})\div 8$. 6. $\frac{2}{3}, \frac{3}{2}, (-3\pm\sqrt{-7})\div 4$. 7. $1, -1, +\sqrt{-1}, -\sqrt{-1}$. 8. $3, -10, (-7\pm 3\sqrt{-11})\div 2$. 9. $7, -28$. 10, $7, -13, \left\{-3\pm\sqrt{79}\right\}$.

Exercises : Page 24

1. 3, 5. 2. 1, -1. 3. 36, 12. 4. 1, 2, 3. 5. $\frac{354}{41}$, $-\frac{120}{41}$, $-\frac{23}{41}$.

Exercises : Page 27

1. 4, 3 ; $9\frac{3}{7}$, $-2\frac{3}{7}$. 2. 5, 3 ; 3, 5. 3. ± 3, ∓ 7 ; $\pm 5\sqrt{-5}$, $\mp 2\sqrt{-5}$. 4. ± 2, ∓ 1 ; $\pm\frac{46}{25}$, $\mp\frac{1}{25}$. 5. 5, 4 ; 4, 5. 6. ± 2, ∓ 1 ; ± 1, ∓ 2. 7. 2, 3 ; $\frac{234}{107}$, $\frac{306}{107}$. 8. ± 7, ∓ 6 ; ± 6, ∓ 7. 9. 9, 12 ; 0, 0. 10. $\frac{5}{2}$, $\frac{3}{2}$; $\frac{3}{2}$, $\frac{5}{2}$; $(-5\pm\sqrt{-35})\div 4$, $(-5\mp\sqrt{-35})\div 4$. 11. 7, 5 ; $(2+12m)\div(1+m^2)$, $(2m+12m^2)\div(1+m^2)$ where $m=(-5\pm\sqrt{14})\div 11$.

Examples : Page 27

1. One real root between -2 and -1, one between 0 and 1 and one between 3 and 4. 4. $51\frac{3}{7}$ c. ft. 7. 2·9, nearly. 8. 5 or $-\frac{1}{3}$. 9. $k=7$, roots 3, 5 ; $k=-1$, roots $+1$, -1. 11. 2, -14, $-6\pm\sqrt{-54}$. 12. $(-3\pm\sqrt{102})\div 2$, $\left\{3, -6\right\}$. 13. 3, $\frac{1}{3}$, $(7\pm\sqrt{13})\div 6$. 14. 1, $\frac{1}{2}(m\pm\sqrt{m^2-4})$ where $m=(-1\pm\sqrt{5})\div 2$.

15. ± 1, $\pm\sqrt{m}$ where $m=\frac{1}{2}(-1\pm\sqrt{-3})$. 16. 3, 5 ; $\frac{21}{17}$, $-\frac{35}{17}$. 17. 1, 5 ; 5, 1 ; 2, 3 ; 3, 2. 18. ± 2, ∓ 3 ; $\pm 8\sqrt{-1}$, $\mp 5\sqrt{-1}$. 19. 0, 0 ; $\frac{12}{7}$, $\frac{4}{7}$; -4, 2. 20. 1, 3 ; 3, 1 ; $2\pm 5\sqrt{-1}$, $2\mp 5\sqrt{-1}$. 22. -14. 23. 20 ct. a doz. 24. 150 mi. 25. Line bisected. 27. Equally.

Examples : Page 37

1. 7 : 4 or 5 : 3. 2. -3. 3. 7. 4. $ab : a'b'$. 5. 30. 17. ± 3, ± 5, ± 11. 18. 0, 0, 0 ; 3, 2, 1. 19. 2, 3, 4.

Exercises : Page 45

1. $a=\frac{22}{7}r^2$. 2. $a=\frac{88}{7}r^2$. 3. $r=6$. 4. 8·16 ft. 5. 223·15 days.

Exercises : Page 50

1. $v=\frac{1}{3}bh$. 2. $v=3{\cdot}1416\,r^2h$; 197·9208. 3. 53·19g. 4. (1) $y=2$; (2) $x=1{\cdot}5$.

Examples : Page 51

1. $v^2=64{\cdot}4s$. 2. 0·0622 *ohms*. 3. $v=\frac{1}{3}bh$. 5. x and y sides of a right-angled triangle remaining similar to itself. 6. $x=14$. 7. $y=12$. 8. $y=1$. 9. $y=10{\cdot}5$. 10. 1·693 m. 11. 3·98125 ft. 12. $x : y :: 3 : 4$ or $:: 4 : 3$.

Exercises : Page 55

1. (1) 93, $2n-1$; (2) 283, $6n+1$; (3) -155, $33-4n$; (4) -504, $107-13n$. 2. $53+48+43+\ldots\ldots$; -57, -2. 3. 3, 12, 21. 4. 30. 11. 69. 13. 0.

Exercises : Page 58

1. 36, 53, ... , 206. **2.** 13, 9, ... , -3. **3.** $\left\{(n-r+1)a+rb\right\}\div(n+1)$. **4.** $13\frac{1}{3}$, $15\frac{2}{3}$, ... , $22\frac{2}{3}$.

Exercises : Page 60

1. (1) 8533 ; (2) $1510\frac{1}{2}$; (3) -689 ; (4) 53 ; (5) $(53a+1378b)\div b$. **2.** (1) $2n^2+7n$; (2) $\frac{1}{2}(51n-5n^2)$; (3) $\frac{1}{8}(5n^2+15n)$. **5.** 4 or 8. **6.** 5. **7.** 8. **8.** 7. **9.** 13. **10.** $\frac{1}{2}(5r^2+19r)$. **11.** $-52\frac{1}{7}-40\frac{4}{7}-29-\ldots\ldots$; 2405. **13.** 10, 465. **14.** $14r+4$. **15.** 5, 8, 11. **16.** $29+35+41+\ldots\ldots$; 14. **17.** $-17-15-13\ldots\ldots$ **19.** $16{\cdot}1t^2$.

Exercises : Page 64

1. (1) $5^8, 5^{n-1}$; (2) $7.2^8, 7.2^{n-1}$; (3) $2\div 3^8, 2\div 3^{n-1}$; (4) $1\div 2^8$, $1\div 2^{n-1}$; (5) $1\div 2^8, (-1)^{n-1}\div 2^{n-1}$. **2.** $a=480, r=\frac{1}{2}$; $480\div 2^{16}$. **3.** 12, 24, 48. **4.** $r=27$. **9.** $1\frac{1}{3}$. **10.** $r=b^2$.

Exercises : Page 66

1. 21, 63, 189. **2.** 35, 245. . . . . **3.** 24, 144. . . . .

Exercises : Page 68

1. (1) $\frac{1}{2}(5^{29}-1)$; (2) $\frac{1}{6}(7-7^{-28})$; (3) $\frac{1}{3}(2+2^{-28})$; (4) $\frac{2}{5}+\frac{2}{5}(\frac{2}{3})^{29}$; (5) $a(1-a^{2n})\div(1-a^2)$. **2.** (1) 4^n-1 ; (2) $12-12\div 2^n$; (3) $\frac{3}{2}-(-1)^n\frac{1}{2}\div 3^{n\ 1}$. **3.** $(x^n-y^n)\div(x-y)$. **5.** $n+2(x^n-1)\div x^n(x-1)+(x^{2n}-1)\div x^{2n}(x^2-1)$. **6.** 9.12.16. **7.** 17, 51, 221 ; $\frac{425}{3}$, $-\frac{595}{3}$, $\frac{833}{3}$. **8.** 48, 72, 108. **10.** 1st term $h(r-1)$; ratio r. **12.** $\frac{10}{9}(10^n-1)-n$. **13.** $\left\{a^2(a^n-1)\div(a-1)-b^2(b^n-1)\div(b-1)\right\}\div(a-b)$.

Exercises : Page 73

1. (1) $\frac{3}{2}$; (2) 3 ; (3) $\frac{2}{3}$. **2.** (1) 15 ; (2) $3\frac{1}{2}$; (3) 21. **5.** $r=\frac{4}{7}$. **6.** $\frac{7}{9}$, $\frac{13}{999}$, $\frac{73146}{99900}$. **7.** Area of original square.

Exercises : Page 76

1. \$1473·17. **2.** \$1484·12. **3.** \$4069·10. **4.** \$3000. **5.** \$1849·34. **6.** \$7860·18.

Exercises : Page 78

1. 105, -105, -35, -21. **2.** 6 and infinity. **3.** 840, 420, 280.

Exercises : Page 80

1. 12, 15. **2.** $7\frac{1}{2}$, 6, $4\frac{4}{5}$. **5.** $pq\div(p+q)$.

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EXERCISES: PAGE 82

1. $n(4n^2-1)\div 3$. 2. $n(n+1)(n+2)\div 3$. 3. $2n(n+1)(2n+1)\div 3$. 4. $n(4n^2+17n+21)\div 2$. 5. $na^2+n(n-1)ab+n(n-1)(2n-1)b^2\div 6$.

EXERCISES: PAGE 84

1. $n^2(2n^2-1)$. 2. $n(n+1)(n+2)(n+3)\div 4$. 3. $2n^2(n+1)^2$. 4. $n(n+1)(9n^2+23n+13)\div 6$. 5. $na^3+\frac{1}{2}n(n+1).3a^2b+\frac{1}{2}n(n+1)(2n+1)ab^2+\frac{1}{4}n^2(n+1)^2b^3$.

EXERCISES: PAGE 85

1. $(1-x^n)\div(1-x)^2-nx^n\div(1-x)$. 2. $6-1\div 2^{n-3}-(2n-1)\div 2^{n-1}$. 3. $7\div(1-x)+5x(1-x^{n-1})\div(1-x)^2-(5n+2)x^n\div(1-x)$. 4. Change x to $-x$ in (1.).

EXAMPLES: PAGE 85

1. (1) 84, 96, 108; (2) $93\frac{3}{4}$, $117\frac{3}{16}$, $146\frac{31}{64}$; (3) 120, 240, infinity. 2. 48, 80. 3. 9, 15, 21, 27. 5. (1) No number if a, b, c are not in A.P.; (2) $(ac-b^2)\div(2b-a-c)$; (3) $(2ac-ab-bc)\div(2b-a-c)$. 7. $5n(n+1)$. 8. $n(n-1)(2n-1)\div 3$. 9. (1) $n-\frac{1}{9}+\frac{1}{9}\div 10^n$; (2) $7n\div 9-7\div 81+7\div(81.10^n)$. 10. $r(r^2+1)\div 2$. 11. $a(1-r^n)\div(1-r)$ where $r^{n+1}=b\div a$. 13. (1) $(1-r^n)\div(1-b)(1-r)-b(1-b^nr^n)\div(1-b)(1-br)$; (2) $x(1-x^n)\div(1-x)+\frac{1}{2}an(n+1)$; (3) $n(n+1)(6n^2-2n-1)\div 6$. 14. $(1-x)^{-2}+2x(1-x^{n-1})(1-x)^{-3}-(n^2+2n-1)x^n(1-x)^{-2}+n^2x^{n+1}(1-x)^{-2}$. 18. $b+c(2r-1)$; an A.P. except for the first term. 19. $4+12+20+\ldots\ldots$ 20. $(a\div r-b\div r^2)(r^{2n}-1)\div(r^2-1)r^{2n-2}$

EXERCISES: PAGE 89

1. (1) 3, 3; (2) 3, 6; (3) 1, 6. 2. (1) 4, 4; (2) 6, 12; (3) 4, 24; (4) 1, 24.

EXERCISES: PAGE 91

1. 120. 2. 14820309504000; 3991680; 40320. 3. 840. 4. 3024; 1680; 336. 5. 4536; 504. 6. 6561. 7. 259. 8. 144. 9. 28800. 10. (1) 362880; (2) 40320; (3) 20160. 11. 2903040. 12. 2880.

EXERCISES: PAGE 93

1. 324632. 2. 35, 126, 1. 3. 455. 4. 18. 5. 35. 6. $n(n-1)(n-2)\div 3!$ 7. 2204475. 8. 14, 90. 9. 21627587520; 152922336. 11. 756756. 12. 126126. 13. 25200. 14. 6300. 16. $\{n(n-1)(n-2)-p(p-1)(p-2)\}\div 3!$ 18. $n(n-1)(n-2)(n-3)\div 8$. 19. 8. 20. 30. 22. 7200; 4320.

Examples : Page 97

1. (1) 15120; (2) 59049 . 210, 1680; 729, 6561. **2.** $(p+1)(q+1)-1$. **3.** 1470, 89. **5.** $(m+n-2)!\div\left\{(m-1)!\,.\,(n-1)!\right\}$. **6.** 800. **7.** $(p+1)!\div\left\{n!\,(p-n+1)!\right\}$. **8.** (1) 495; (2) 35; (3) 10.

Exercises : Page 100

3. 10, 16, 18.

Exercises : Page 102

2. $6!\ x^3y^3\div(3!\ 3!)$; $20!\ x^{10}y^{10}\div(10!\ 10!)$; $-14!\ (2a)^7(3y)^7\div(7!\ 7!)$; $77x^6y^6\div3888$. **3.** $5!\ (2a)^3(3b)^2\div(3!\ 2!)$, $-5!\ (2a)^2(3b)^3\div2!\ 3!$; $-23!\ x^{12}y^{11}\div(12!\ 11!)$, $23!\ x^{11}y^{12}\div(11!\ 12!)$; $17!\ x^{18}y^{16}\div9!\ 8!$, $-17!\ x^{16}y^{18}\div8!\ 9!$. **4.** $(-1)^rn!\ x^r\div(n-r)!\ r!$; $m!\ 3^rx^r\div(m-r)!\ r!$; $(-1)^rn!\ a^{n-r}2^rx^r\div(n-r)!\ r!$

Exercises : Page 108

1. (1) x^r; (2) $(-1)^rx^r$; (3) $(r+1)x^r$; (4) $(r+1)(r+2)x^r\div1.2$; (5) $(r+1)(r+2)(r+3)x^r\div1.2.3$; (6) $m(m+1)\ldots\ldots(m+r-1)x^r\div1.2.3\ldots\ldots r$. **2.** $-1.3.\ldots\ldots(2r-3)x^r\div(2.4.\ldots\ldots2r)$; $1.3.\ldots\ldots(2r-1)x^r\div(2.4.\ldots\ldots2r)$; $(-1)^r\,1.3.\ldots\ldots(2r-5)3a^{\frac{3}{2}}x^r\div(2.4.\ldots\ldots2r.a^r)$; $5.6.\ldots\ldots(r+4)(2a)^{-5}3^rx^r\div(1.2.\ldots\ldots r.2^ra^r)$. **3.** $-9.7.5.3x^6\div2.4.\ldots\ldots12$. **5.** $(-1)^{r-1}2.5.\ldots\ldots(3r-4)x^r\div3.6.\ldots\ldots3r$; $(-1)^r2.5.\ldots\ldots(3r-1)x^r\div r!$; $1.6.11.\ldots\ldots(5r-4)x^r\div(5.10.\ldots\ldots5r)$. **6.** $(r+1)^2$; $2r^2+2r+1$; $8r-4$. **8.** $(-1)^nn(n+1).\ldots\ldots(n+r-1)\div r!$. **10.** (1) $(1-\frac{1}{3})^{-2}$; (2) $(1+\frac{1}{2})^{-\frac{1}{2}}$; (3) $(1-\frac{2}{3})^{-\frac{1}{2}}$; (4) $(1-\frac{3}{5})^{-\frac{1}{3}}$.

Exercises : Page 110

1. 4·89898; 8·94427; 24·8998; 14·9666; 34·9285. **2.** 3·97906; 9·84886; 0·14586; 2·99256; 0·198945. **3.** 0·000026. **4.** $1-x$; $\frac{1}{2}+\frac{23}{16}x$; $\frac{1}{2}+\frac{41}{336}x$; $1-\frac{1}{2}x$; $\frac{14}{45}-\frac{413}{675}x$.

Exercises : Page 116

1. (1) 6th and 7th; (2) 1st; (3) 4th, 6th and 7th; (4) 3rd and 4th. **2.** (1) 5th and 6th; (2) 8th; (3) 24th and 25th; (4) 19th.

Examples : Page 117

1. $(-1)^n\,(2n)!\div(n!\ n!)$. **5.** 2^n; $2^n\,(n+r+1)-n2^{n-1}$. **6.** (1) $\sqrt{5}\div2$; $3^{\frac{1}{3}}\div2^{\frac{1}{3}}-1$. **7.** $2^n\,(n+r)-n2^{n-1}$. **10.** $x^n\div(1-x)$; $\left\{(n+1)\,x^n-nx^{n+1}\right\}\div(1-x)^2$. **12.** $25!\div(5!\,.\,7!\,.\,13!)$. **14.** $n(n-1)\,(4n^2+16n-21)\div6$.